Problems to Enrich and Challenge: New York State Mathematics League Contests 2011–2019

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Preface

In the spring of 1992, I was selected to the New York City math team that traveled to Ithaca for the New York State Math League championships. The selection was mostly due to my performance on citywide competitions throughout the year; I had done well enough to play on the NYC B team and we did pretty well that day.

Before I left for Ithaca, my coach, Harry Rattien, talked with me about how he was looking forward to the "little fish in a big pond" phenomenon that I would experience over the NYSML weekend. He explained that I had done very well at Townsend Harris High School and that (particularly because THHS is a humanities magnet school) I was doing better than my classmates, but I was about to take a trip to a statewide competition with a bunch of my peers and my game would need to rise to the occasion.

I felt completely unprepared for NYSML1992. It kicked my butt.

What NYSML1992 did for me was noteworthy. I was frustrated by mathematics for really the first time, and I knew that I would have to learn so much more mathematics and make so many more mistakes as I tried to understand higher and higher concepts. That Saturday in Ithaca helped to form the mentality that I have taken with me into my math team coaching life, into my pedagogy (for all courses, not just honors courses), and into my life as a whole.

The New York State Mathematics League was founded in 1973 by Alfred Kalfus, whose vision was to have students from across the state gather for face-to-face competition annually. As a result, thousands of students have come together to compete over the years, deepening existing friendships while forging new ones, and working individually and corporately on some of the most challenging problems available to high school students.

NYSML moves from location to location each year as different member leagues take responsibility for hosting. Some of our member leagues hail from the I-90 stretch of the New York State Thruway (the Monroe County Math League, the Onondaga County Math League, and the Albany Area Math Circle). Others are from the more central part of New York (the Genesee Valley Math League, the Ithaca High School math team, and the Southern Tier Interscholastic Math League). There is a fair concentration of teams from the I-87 corridor close to New York City (the Duchess-Ulster-Sullivan-Orange Math League (DUSO), the Rockland County Math League, and the Westchester-Putnam Math League). Teams also come to NYSML from the New York City / Long Island area (the New York City Math Team, the Nassau County Interscholastic Math League, and the Suffolk County Math League). As always, NYSML is looking for more leagues to join, and we welcome suggestions that would make our membership more representative of the state as a whole.

The contest begins with a Team Round, where teams of 15 students work collaboratively to solve ten problems in 20 minutes. The teams then work collaboratively on the Power Question, in which the questions revolve around a central theme and results are proven with rigor. The Individual Round follows; students answer ten questions in five pairs, taking ten minutes for each pair. The last round is the Relay Round, in which sub-teams of three try to answer a string of questions, where the answer to the first question is needed to solve the second, and the answer to the second question is needed to solve the third; only the third person's answer is scored.

Those four rounds are the only rounds that count toward overall team and individual results. In the Team Round, each correct answer earns 5 points. It is not unusual for the top teams to earn 45 or 50 points in the round, but the average for all teams has been about 25 points. The Power Question is worth 50 points, and scoring in this round runs the gamut from teams earning almost all of the points to teams earning very few points. Each Individual problem is worth 1 point per contestant, meaning that the team can earn as many as $10 \cdot 15 = 150$ points on this round. Very few teams earn more than 100 of those 150 points. On the Relay Round, only the third person's answer is scored. If the group of 3 gets the problem correct within 3 minutes they earn 5 points, and a correct answer within 6 minutes earns 3 points. Thus, each relay is worth a maximum of $5 \cdot 5 = 25$ points; most teams end up scoring about half of the possible number of points for the round.

The Tiebreaker Round follows the relays. Students with high Individual Round scores come to the front of the auditorium and answer questions one at a time, using their times to break ties and award final prizes. The Individual Champion earns the Curt Boddie Award in memory of Curt, who was NYSML's President for many years.

NYSML has an executive board that oversees the contest. I currently serve as President. Mike Curry is our Executive Director. Our Vice President is Anchala Sobrin, and our Treasurer is Kim Dwyer. We serve at the pleasure of the league, and our joy is in seeing the students enjoy mathematical competition on a spring weekend each year.

I became head author in 2010, after the death of Dr. Leo J. Schneider, who had been NYSML's main author since 2000. My first job was to ensure that NYSML2011, which had been basically written by Leo, was edited and brought to print in time for competition. After NYSML2011, we decided to use a committee approach to help mitigate against biases in topic areas that might be present if there were a small number of authors. The current author committee includes Matthew Babbitt, Stan Kats, Jason Mutford, Dick Olson, and Tom Weisswange. Questions are proposed in the late summer. Drafts are sent to the committee in October. Solutions are written and edited and rewritten and reedited. In recent years, Chris Jeuell has helped with the final editing process. I will also note that some of the Power Questions in this book were authored by Dave Phillips, and the 2015 Power Question was authored by Oleg Kryzhanovsky and Stan Kats. NYSML is grateful to every person who has contributed to our contests over the years.

This book was assembled using files that Chris Jeuell created. We are grateful for the way Chris continues to give of himself for the benefit of the mathematics community.

The NYSML executive board is proud of the work we do, and we hope this book helps you to grow as mathematicians and as problem solvers. Enjoy!

George R. Reuter, Jr., President of NYSML

2011 Contest at Suffolk County Community College (Suffolk)

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2011 Team Problems

T-1. Suppose that a small state which has been issuing automobile license plates consisting of four digits from 0000 through 9999 decides they need more plates. So, they discontinue their four-digit plates in favor of four-letter plates from $AAAA$ through $ZZZZ$. Compute the number of additional plates available when using four letters instead of four digits.

T-2. Let x, y, and z be digits in the base three numeral system with $x \neq 0$. Compute the sum of all n for which $n = 5 \cdot xyz_{\text{base three}} = 2 \cdot xyz_{\text{base five}}$

T-3. Compute x so

$$\frac{\log_2(\log_3(\log_4(x)))}{\log_4(\log_3(\log_2(x)))} = 0.$$

T-4. Compute the number of triples (x, y, z) of positive integers for which $4x + 4y + z = 2011$.

T-5. Let $A(7,1,2), B(13,9,2), C(1,18,2), D(7,1,12), E(13,9,12), F(1,18,24)$ be the six points in x-y-z-space that are the vertices of a convex polyhedron $\mathcal{P}$. Compute the volume of $\mathcal{P}$.

T-6. Let $f(x) = x^2 + bx + c$ for constants b and c, let $g(x) = f(f(x))$, and suppose that the graph of $y = g(x)$ is symmetric about the y-axis. Given that the y-intercepts of the graphs of $y = f(x)$ and $y = g(x)$ differ by 60 and one is positive and the other is negative, compute the y-intercept of the graph of $y = g(x)$.

T-7. Planar polygon $ABCD$ is a kite suspended in xyz-space, with $A = (4,5,9)$, $B = (10,11,17)$, and $C = (20,11,23)$. A plane $\mathcal{P}$ perpendicular to the xy-plane contains diagonal $\overline{BD}$. An (x', y')-coordinate system is imposed on plane $\mathcal{P}$ with the same unit of measure as in that used in the xyz-coordinate system. The x'-axis is parallel to the xy-plane, and the y'-axis is parallel to the z-axis. In the (x', y')-coordinate system, the equation of the line containing $\overline{BD}$ is $y' = mx'$ with $m > 0$. Compute m.

T-8. Each edge of cube $\mathcal{C}$ measures 2011. Line ℓ is perpendicular to a face of $\mathcal{C}$ at its center. The cube is rotated around ℓ, and $\mathcal{R}$ is the set of all points touched by $\mathcal{C}$ in its rotation. Sphere $\mathcal{S}$ is the smallest sphere containing $\mathcal{C}$. Let C, R, and S be the volumes of $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{S}$, respectively. Compute $\dfrac{S - R}{C}$.

T-9. In $\triangle ABC$, $AB = 24$, $AC = 10$, and $BC = 26$. Points D_i are on side $\overline{AC}$, and $AD_i = i$ for $i = 1, 2, \ldots, 9$. Points E_i are on $\overline{BC}$ and $\overline{D_iE_i} \perp \overline{AC}$ for $i = 1, 2, \ldots, 9$. Points P_i are on $\overline{D_iE_i}$ so that $\dfrac{D_iP_i}{D_iE_i} = \dfrac{AD_i}{AC}$ for $i = 1, 2, \ldots, 9$. Compute $\displaystyle\sum_{i=1}^{9} D_iP_i$.

T-10. When n is a positive even integer, a deck of n cards numbered 1 through n can be given a perfect out-riffle or a perfect in-riffle by dividing the deck into halves, top and bottom, and then interleaving the cards, with the top card in the original order going on top in the out-riffle and going second from the top with the in-riffle. For example, when $n = 6$, the results of an out-riffle and followed by an in-riffle are shown:

Original Order	**Out−Riffle the Original**	**In−Riffle the Outriffle**
1	$1 \Rightarrow \mathbf{1}$	$5 \Rightarrow \mathbf{5}$
2	$4 \Rightarrow \mathbf{4}$	$1 \Rightarrow \mathbf{1}$
3	$2 \Rightarrow \mathbf{2}$	$3 \Rightarrow \mathbf{3}$
4	$5 \Rightarrow \mathbf{5}$	$4 \Rightarrow \mathbf{4}$
5	$3 \Rightarrow \mathbf{3}$	$6 \Rightarrow \mathbf{6}$
6	$6 \Rightarrow \mathbf{6}$	$2 \Rightarrow \mathbf{2}$

Let $n = 52$ with the original order being an increasing order from 1 through 52. The deck then undergoes a long sequence of an equal number of out-riffles and in-riffles in some random order. Afterwards, card number 22 is in the 47^{th} position from the top and card 31 is in the k^{th} position from the top. Compute all possible values of k.

2011 Team Answers

T-1. 446976

T-2. 180

T-3. 64

T-4. 125751

T-5. 1050

T-6. $60 - 2\sqrt{15}$

T-7. $\dfrac{1}{\sqrt{13}}$ or $\dfrac{\sqrt{13}}{13}$

T-8. $\dfrac{\sqrt{3}\pi}{2} - \dfrac{\pi}{2}$ or equivalent simplified result

T-9. 39.6 or $39\frac{3}{5}$ or $\dfrac{198}{5}$

T-10. 6

2011 Team Solutions

T-1. The answer is $26^4 - 10^4$. Use any of the following computational trails to the answer.
One could notice that $26^4 - 10^4 = 456976 - 10000 = \mathbf{446976}$.
One could also notice that $26^4-10^4 = (26^2-10^2)(26^2+10^2) = (26-10)(26+10)(4)(169+25) = 16 \cdot 36 \cdot 4 \cdot 194 = 446976$.

T-2. Since $5(9x+3y+z) = 2(25x+5y+z)$, we have $3z = 5(x-y)$ so z must be a multiple of 5 and $(x-y)$ must be a multiple of 3. Since $x, y, z \in \{0,1,2\}$, it follows that $z = 0$ and $x = y$. Hence the possible values of n are $n = 5 \cdot 110_{\text{base three}} = 5 \cdot 12 = 60 = 2 \cdot 30 = 2 \cdot 110_{\text{base five}}$ and $n = 5 \cdot 220_{\text{base three}} = 5 \cdot 24 = 120 = 2 \cdot 60 = 2 \cdot 220_{\text{base five}}$. Their sum is $60 + 120 = \mathbf{180}$.

T-3. A fraction equals zero provided its numerator equals zero and its denominator is defined and does not equal zero.

$$\log_2(\log_3(\log_4(x))) = 0 \iff \log_3(\log_4(x)) = 1 \iff \log_4(x) = 3 \iff x = 4^3 = \mathbf{64}$$

We need to check that for $x = 4^3 = 2^6$, the denominator is defined and not zero. Note that

$$\log_4(\log_3(\log_2(2^6))) = \log_4(\log_3(6)) \text{ and } 1 = \log_3(3) < \log_3(6) < \log_3(9) = 2,$$

so $0 = \log_4(1) < \log_4(\log_3(6)) < \log_4(2) = 1/2$ and therefore the denominator is defined and non-zero.

T-4. Since for each pair of positive integers (x, y) there will be at most one positive integer z so that $4x+4x+z = 2011$, we need only count the number of ordered pairs (x, y) for which there is a positive integer z satisfying the equation. If $4x+4y+z = 2011$, then $x+y = 502+\frac{3-z}{4} \le 502$. The number of pairs of positive integers (x, y) for which $x+y = n$ is $n-1$ for $n = 1, 2, \ldots, 502$, so our answer is $1 + 2 + \cdots + 501 = \frac{(1+501)(501)}{2} = \mathbf{125751}$.

T-5. The polyhedron can be partitioned into a prism $\mathcal{Q}$ with faces $\triangle ABC$ and $\triangle DEG$ and a tetrahedron $\mathcal{T}$ with base $\triangle DEG$ and vertex F where G is the point $(1, 18, 12)$. In the $z = 2$ plane, the slopes of $\overline{AB}$ and $\overline{BC}$ are $\frac{9-1}{13-7} = \frac{4}{3}$ and $\frac{18-9}{1-13} = -\frac{3}{4}$, so $\overline{AB} \perp \overline{BC}$. Therefore the area of $\triangle ABC$ is $\frac{1}{2}(AB)(BC) = \frac{\sqrt{(9-1)^2+(13-7)^2} \cdot \sqrt{(1-13)^2+(18-9)^2}}{2} = 75$, and the volume of $\mathcal{Q}$ is $75(12-2) = 750$. Since $\triangle DEG \cong \triangle ABC$, the volume of $\mathcal{T}$ is $\frac{1}{3}(75)(24-12) = 300$. Consequently the volume of $\mathcal{P}$ is $750 + 300 = \mathbf{1050}$.

T-6. Since $g(x) = (x^2+bx+c)^2+b(x^2+bx+c)+c = x^4+2bx^3+(b^2+2c+b)x^2+(2bc+b^2)x+(c^2+bc+c)$ is symmetric about the y-axis, the coefficients of x^3 and x must be zero, so $b = 0$. The difference between the y-intercepts is $g(0) - f(0) = (c^2 + c) - c = c^2 = 60$, so $c = -2\sqrt{15}$ since the intercepts differ in sign. Thus, $g(0) = c^2 + c = \mathbf{60 - 2\sqrt{15}}$.

T-7. The diagonals of any kite intersect at right angles, and since

$$AB = \sqrt{(10-4)^2 + (11-5)^2 + (17-9)^2} = \sqrt{136} = \sqrt{(20-10)^2 + (11-11)^2 + (23-17)^2} = BC,$$

it follows that the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at the midpoint $E = (12, 8, 16)$ of $\overline{AC}$. Compute m from the slope of the line segment $\overline{BE}$. Since plane P is perpendicular to the $\frac{\Delta y'}{\Delta x'} = \frac{17-16}{\sqrt{(12-10)^2+(8-11)^2}} = \mathbf{\frac{1}{\sqrt{13}}}$.

T-8. For convenience in computation, let $\mathcal{C}$ have side s. Then $\mathcal{R}$ is a cylinder of radius $r = \frac{\sqrt{2}}{2}s$ and height $h = s$, and the radius of $\mathcal{S}$ is $\rho = \frac{\sqrt{3}}{2}s$. Consequently,

$$\frac{S-R}{C} = \frac{\frac{4}{3}\pi\rho^3 - \pi r^2 h}{s^3} = \frac{\frac{4}{3}\pi\left(\frac{\sqrt{3}}{2}s\right)^3 - \pi\left(\frac{\sqrt{2}}{2}s\right)^2 s}{s^3} = \mathbf{\frac{\sqrt{3}\pi}{2} - \frac{\pi}{2}}.$$

T-9. Let $A = (0,0)$, $B = (0,24)$, and $C = (10,0)$. Then $D_i = (i, 0)$, $E_i = (i, 24-2.4i)$, and

$$\begin{aligned}\sum_{i=1}^{9} D_iP_i &= \sum_{i=1}^{9} \frac{D_iE_i \cdot AD_i}{AC} = \sum_{i=1}^{9} \frac{(24-2.4i)\cdot i}{10} \\ &= \sum_{i=1}^{9} 2.4i - 0.24i^2 = 2.4\frac{(9)(10)}{2} - 0.24\frac{(9)(10)(19)}{6} = \mathbf{39.6}.\end{aligned}$$

T-10. Note that after every out-riffle and in-riffle, cards that were symmetric with respect to the center remain symmetric to the center. Since cards numbered 22 and 31 start out symmetric with respect to the center, they will remain symmetric to the center. Since card number 22 in position 47 is in the 6th position from the bottom of the deck, card number 31 will be in the **6**th position from the top.

2011 Individual Problems

I-1. Let $x_1 = 1000$, $x_2 = 1001$, $x_3 = 1002$, $x_4 = 1010$, $\dots$, $x_{54} = 2222$ be the increasing sequence of all four-digit integers that can be written using only the digits 0, 1, and 2. Compute j so that $x_j = 2011$.

I-2. If r is chosen at random from $\{0, 1, 2, \dots, 10\}$, compute the probability that there exists an integer x so that x^2 leaves a remainder of r when divided by 11.

I-3. Compute the ordered triple (a, b, c) of base 12 digits if $\sqrt{14641_{\text{base 16}}} = abc_{\text{base 12}}$. (Use $0, 1, 2, \dots, 9, T, E$ as the twelve base 12 digits.)

I-4. A round picture is centered in a square frame. The visible region inside the square frame has area A. The area of the region that is inside the frame and covered by the picture is equal to the area of the region that is outside the picture but inside the frame. The smallest square frame that will enclose the circular picture has a visible region of area B. Compute B/A.

I-5. A polyhedron has exactly 8 triangular faces and exactly 6 octagonal faces. Compute the number of corners the polyhedron has.

I-6. A fair coin is flipped 50 times, and the result is 25 heads and 25 tails. Let p be the probability that during these 50 flips the difference between the number of heads and the number of tails is never larger than 1. When $p = 2^\alpha 3^\beta \cdots$ is written as a product of distinct primes to integer powers, compute α, the power of 2.

I-7. Compute the quadruple (w, x, y, z) for which

$$\begin{aligned}
2 \cdot w + 0 \cdot x + 1 \cdot y + 1 \cdot z &= 2 \\
1 \cdot w + 2 \cdot x + 0 \cdot y + 1 \cdot z &= 0 \\
1 \cdot w + 1 \cdot x + 2 \cdot y + 0 \cdot z &= 1 \\
0 \cdot w + 1 \cdot x + 1 \cdot y + 2 \cdot z &= 1.
\end{aligned}$$

I-8. For $n \geq 3$, let $f(n)$ be the number of subsets of three elements that can be chosen from a set of n distinct elements. Compute

$$\sum_{n=3}^{101} \frac{1}{f(n)}.$$

I-9. The first quadrant of the xy-plane is painted in horizontal stripes, alternating red and blue. The stripes are 1 unit wide and the lowest stripe is red. Compute the sum of the lengths of the intervals on the x-axis, for $0 < x < 12345$, where the graph of $y = \log_{10}(x)$ is in a blue stripe.

I-10. A circle of radius r is inscribed in quadrilateral $ABCD$, and $AB = 8$, $CD = 18$, $\overline{AB} \parallel \overline{CD}$, and $BC = AD$. Compute r.

2011 Individual Answers

I-1. 32

I-2. $\dfrac{6}{11}$

I-3. $(2, 0, 1)$

I-4. $\dfrac{2}{\pi}$

I-5. 24

I-6. 21π

I-7. $\left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

I-8. $\dfrac{15147}{10100}$

I-9. 9090

I-10. 6

2011 Individual Solutions

I-1. There are $3^3 = 27$ numbers $1abc$ with $a, b, c \in \{0, 1, 2\}$, so $x_{28} = 2000$ and $x_{27+i} = 2abc$ when $x_i = 1abc$. Since $x_5 = 1011$, it follows that $x_{32} = 2011$. The answer is **32**.

Alternate Solution: Half of the 54 x_i begin with 1 and half with 2, so $x_{27} = 1222$, $x_{28} = 2000$, $x_{29} = 2001$, $x_{30} = 2002$, $x_{31} = 2010$, and $x_{32} = 2011$. The answer is 32.

Alternate Solution: Note that the increasing sequence of positive integers written in base 3 notation is precisely the same, in notation, as the sequence of base 10 integers that can be written with the digits $\{0, 1, 2, \}$. To find the position of 2011 among the four-digit numbers, we subtract the number of positive integers with three or fewer digits:

$$2011_{\text{base three}} - 222_{\text{base three}} = [(2)3^3+(1)3+1]-[(2)3^2+(2)3+2] = 58_{\text{base ten}} - 26_{\text{base ten}} = 32_{\text{base ten}}.$$

Alternate Solution: The sequence can be thought of as four-digit numbers in base 3. Since $x_1 = 1000_3 = 1 \cdot 3^3 + 0 \cdot 3^2 + 0 \cdot 3 + 0 \cdot 1 = 27$ and $2011_3 = 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3 + 1 = 58$, then $i = 58 - 27 + 1 = 32$.

I-2. Since the remainders of x^2 and $(x + 11)^2$ are the same when divided by 11, we check the remainders for 0^2 through 10^2. Notice: $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 11 + 5$, $5^2 = 2(11) + 3$, $6^2 = 3(11) + 3$, $7^2 = 4(11) + 5$, $8^2 = 5(11) + 9$, $9^2 = 7(11) + 4$, $10^2 = 9(11) + 1$. Thus one can see that the only possible remainders are $0, 1, 4, 9, 5, 3$. The answer is $\mathbf{\frac{6}{11}}$.

Alternate Solution: Note that only 0^2 leaves a remainder of 0; and that for $0 < x < 11$, the numbers x^2 and $(11 - x)^2$ leave the same remainder when divided by 11. To see that there are five non-zero remainders, we need simply check that the remainders for $1^2, 2^2, \ldots, 5^2$ are all distinct. As in the main solution, the answer is $\frac{6}{11}$.

I-3. For any base $b > 6$, $\sqrt{14641_{\text{base } b}} = \sqrt{b^4 + 4b^3 + 6b^2 + 4b + 1} = (b + 1)^{4/2}$. Let $b = 16$. Then $(16 + 1)^2 = 289 = 288 + 1 = 2{\cdot}12^2 + 1 = 201_{\text{base } 12}$. The answer is $(\mathbf{2}, \mathbf{0}, \mathbf{1})$.

I-4. The large frame is $s \times s$, the round picture has radius r, and the small frame is $2r \times 2r$. Since $\pi r^2 = s^2 - \pi r^2$, we have $s^2 = 2\pi r^2$. Hence $\dfrac{B}{A} = \dfrac{(2r)^2}{s^2} = \dfrac{4r^2}{2\pi r^2} = \dfrac{\mathbf{2}}{\pi}$.

I-5. Use Euler's Formula: $F - E + V = 2$. We are given that $F = 8 + 6 = 14$. Each edge of the polyhedron is formed by two sides of the polygons being adjacent, so $E = \frac{8 \cdot 3 + 6 \cdot 8}{2} = 36$. Thus $V = 2 + E - F = 2 + 36 - 14 = \mathbf{24}$.

Alternate Solution: The statement of the problem implies that all polyhedra with 8 triangular faces and 6 octagonal faces have the same number of vertices. Therefore, imagine starting with a cube and cutting a small tetrahedron from each corner. Each of the six original faces of the cube becomes an octagon, and there is a small triangle where each of the eight corners of the cube were. The resulting polyhedron has 3 vertices where each of the 8 vertices of the cube used to be. The answer is $3 \cdot 8 = 24$.

I-6. Note that if the difference between the number of heads and the number of tails is never greater than 1, then each even numbered toss must always be the opposite of the odd numbered toss that precedes it. Hence the 50 tosses can be partitioned into 25 pairs that are one of HT or TH. Thus there are 2^{25} sequences of heads and tails in which the difference between the number of heads and tails is never larger than 1. There are $\binom{50}{25}$ ways to arrive at 25 heads and 25 tails because any subset of the 50 flips can be the 25 heads. Therefore

$$p = \frac{2^{25}}{\binom{50}{25}} = \frac{2^{25}}{\frac{50\cdot 49\cdot 48\cdot \ldots \cdot 26}{25\cdot 24\cdot \ldots \cdot 13\cdot 12\cdot 11\cdot 10\cdot \ldots \cdot 2\cdot 1}}.$$

Note that every even term in $50 \cdot 49 \cdot 48 \cdot \cdots \cdot 26$ will cancel, leaving thirteen factors of 2 in the numerator, thereby making the numerator of the fraction equal to 2^{12}. But from $12 \cdot \ldots \cdot 1$ we pick up 10 factors of 2 from the terms 12, 10, 8, 6, 4, and 2, giving $2^{12+10} = 2^{22}$ as the numerator. The answer is **22**.

Alternate Solution: Note that $p = \frac{2^{25}}{\binom{50}{25}}$. Because $\binom{50}{25} = \frac{50!}{(25!)^2}$, it follows that

$$\alpha = 25 - \left(\sum_{i=1}^{\infty} \left\lfloor \frac{50}{2^i} \right\rfloor - 2\sum_{i=1}^{\infty} \left\lfloor \frac{25}{2^i} \right\rfloor \right) = 25 - (25 + 12 + 6 + 3 + 1) + 2(12 + 6 + 3 + 1)) = 22.$$

I-7. Sum the four equations and divide through by 4 to see that $w+x+y+z=1$. Then subtract that successively from each of the four equations to get $w-x=1$, $x-y=-1$, $y-z=0$, and $z-w=0$, from which it follows that $y=z=w$. Substitution into the first given equation shows that $y=z=w=\frac{1}{2}$, and since $w-x=1$, we have $x=-\frac{1}{2}$. The answer is $\left(\mathbf{\frac{1}{2}}, -\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)$.

I-8. Since $\dfrac{1}{f(n)} = \dfrac{1}{\binom{n}{3}} = \dfrac{3\cdot 2\cdot 1}{(n-2)(n-1)n} = \dfrac{3}{n-2} - \dfrac{6}{n-1} + \dfrac{3}{n}$ we have

$$\sum_{n=3}^{101} \frac{1}{f(n)} = \left(\frac{3}{1} - \frac{6}{2} + \frac{3}{3}\right) + \left(\frac{3}{2} - \frac{6}{3} + \frac{3}{4}\right) + \left(\frac{3}{3} - \frac{6}{4} + \frac{3}{5}\right) + \cdots + \left(\frac{3}{99} - \frac{6}{100} + \frac{3}{101}\right).$$

Note that when all the terms with the same denominators are combined, the terms with denominators $3, 4, \ldots, 99$ have numerators $3 - 6 + 3 = 0$, so the sum telescopes into

$$\left(\frac{3}{1} - \frac{6}{2}\right) + \frac{3}{2} + \frac{3}{100} + \left(-\frac{6}{100} + \frac{3}{101}\right) = \frac{3}{2} - \frac{3}{100} + \frac{3}{101} = \frac{15147}{100\cdot 101}.$$

The answer is $\frac{\mathbf{15147}}{\mathbf{10100}}$.

I-9. The stripes are blue for $n < y < n+1$ when n is odd. Since $\lfloor \log(x) \rfloor$ is odd when $10^n \le x < 10^{n+1}$ for n odd, the desired intervals are $10 \le x < 10^2$ and $10^3 \le x < 10^4$ for a total length of $(100-10)+(10000-1000) = \mathbf{9090}$.

I-10. By examining the lengths of the segments from the vertices to the points of tangency one observes that sums of opposite sides of quadrilaterals with inscribed circles are equal. Thus $2AD = AD + BC = AB + CD = 8 + 18 = 26$, so $AD = BC = 13$. Since $\overline{AB} \parallel \overline{CD}$ and $BC = AD$, the quadrilateral is an isosceles trapezoid. Let E be the foot of the altitude from A to $\overline{CD}$. Then $DE = \frac{1}{2}(CD - AB) = 5$, so $AE = \sqrt{AD^2 - DE^2} = \sqrt{13^2 - 5^2} = 12$. The height of the trapezoid is the diameter of the circle, so $r = \frac{12}{2} = \mathbf{6}$.

Power Question 2011: Colorings

To *Vertex-color* a polygon or polyhedron means to assign colors to the vertices so that no edge joins vertices of the same color. To *Edge-color* a polygon or polyhedron means to assign colors to the edges so that no two edges of the same color meet at a vertex. To *Face-color* a polyhedron means to assign colors to the faces so that no edge is adjacent to a pair of faces of the same color.

Example: In how many ways can the vertices of the pentagon $ABCDE$ be Vertex-colored using at most six colors?

Discussion: We can choose the color for A in 6 ways, then for B use any of the 5 colors not used for A, then for C use any of the 5 colors not used for B, then for D use any of the 5 colors not used for C, but when we come to color E we don't know whether we can use 4 or 5 colors because the choice depends on whether D and A are the same or different colors. This method yields the estimate that the number of colorings will be between $6 \cdot 5 \cdot 5 \cdot 5 \cdot 4$ and $6 \cdot 5 \cdot 5 \cdot 5 \cdot 5$; i.e., between 3000 and 3750. To get an accurate count of the number of Vertex-colorings, we can keep track of whether or not each vertex is colored the same as the first vertex, as in the following solution.

Solution: Choose any of the 6 colors for A and then any of the 5 remaining colors for B.
If we choose C to be the same color as A, then D will have to be a color different from A, and E must be one of the 4 colors not used for A or D. Consequently, in this case the product of the number of ways to color A, B, C, D, E in that order will be $6 \cdot 5 \cdot 1 \cdot 5 \cdot 4 = 600$.
If we choose C to be a color different from A (and also different from B), then there are two possibilities for D: (1) D and A the same color; (2) D and A different colors. Thus, in the case that C and A are different colors, the number of ways to color A, B, C, D, E is the sum $6 \cdot 5 \cdot 4 \cdot 1 \cdot 5 + 6 \cdot 5 \cdot 4 \cdot 4 \cdot 4 = 2520$.
So the total number of ways to Vertex-color $ABCDE$ will be $600 + 2520 = 3120$.

P-1. **Coloring Polygons with a Minimum Number of Colors**

a. Compute the ordered quadruple (v_3, v_4, v_5, v_6) where v_n is the smallest number of colors that can be used to Vertex-color a polygon with n sides. [4 pts]

b. *(i)* Give a general rule or formula for v_n, the smallest number of colors that can be used to Vertex-color a polygon with n sides. *(ii)* Prove your answer. [3 pts]

c. Let v_n be as above and e_n be the smallest number of colors that can be used to Edge-color a polygon with n sides. *(i)* Compute $e_n - v_n$. *(ii)* Prove your answer. [3 pts]

P-2. **Coloring Cubes with a Minimum Number of Colors**

a. *(i)* Compute the minimum number of colors needed to Vertex-color a cube. *(ii)* Explain or illustrate this minimum Vertex-coloring. [4 pts]

b. *(i)* Compute the minimum number of colors needed to Edge-color a cube. *(ii)* Explain or illustrate this minimum Edge-coloring. [3 pts]

c. *(i)* Compute the minimum number of colors needed to Face-color a cube. *(ii)* Explain or illustrate this minimum Face-coloring. [3 pts]

P-3. **Coloring Pyramids with a Minimum Number of Colors**
Let $n \geq 3$. If $A_1A_2 \ldots A_n$ is a regular n-gon and B is a point on the line perpendicular to the center of the n-gon not in the plane of the n-gon, then we will call the polyhedron with edges those of the n-gon together with the segments $\overline{A_iB}$ for all i, a "*pyramid based on an n-gon.*"

a. *(i)* Compute the minimum number of colors needed to Vertex-color a pyramid based on an n-gon. *(ii)* Explain or illustrate this minimum Vertex-coloring. [3 pts]

b. *(i)* Compute the minimum number of colors needed to Edge-color a pyramid based on an n-gon. *(ii)* Explain or illustrate this minimum Edge-coloring. [4 pts]

c. *(i)* Compute the minimum number of colors needed to Face-color a pyramid based on an n-gon. *(ii)* Explain or illustrate this minimum Face-coloring. [3 pts]

P-4. **Coloring Prisms with a Minimum Number of Colors**
Let $n \geq 3$. If $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$ are congruent regular n-gons in parallel planes oriented so the lines $\overline{A_iB_i}$ are perpendicular to the planes of the n-gons for all i, call the polyhedron with edges those of the two n-gons together with the segments $\overline{A_iB_i}$ for all i a "*prism based on an n-gon.*"

a. *(i)* Compute the minimum number of colors needed to Vertex-color a prism based on an n-gon. *(ii)* Explain or illustrate this minimum Vertex-coloring. [3 pts]

b. *(i)* Compute the minimum number of colors needed to Edge-color a prism based on an n-gon. *(ii)* Explain or illustrate this minimum Edge-coloring. [4 pts]

c. *(i)* Compute the minimum number of colors needed to Face-color a prism based on an n-gon. *(ii)* Explain or illustrate this minimum Face-coloring. [3 pts]

P-5. **Vertex-coloring with k colors**
Let $F_n(k)$ be a formula for the number of Vertex-colorings of a convex n-gon using at most k colors. *Express all of the polynomial answers either multiplied out in decreasing powers of k or factored into linear factors and quadratic factors that have no real roots.*

a. *(i)* Compute $F_3(k)$. *(ii)* Explain how you derived the formula. [2 pts]

b. *(i)* Compute $F_4(k)$. *(ii)* Explain how you derived the formula. [2 pts]

c. *(i)* Compute $F_5(k)$. *(ii)* Explain how you derived the formula. [2 pts]

d. *(i)* Compute the ordered pair of linear polynomials, $(g(k), h(k))$ so $F_n(k) = g(k) \cdot F_{n-1}(k) + h(k)F_{n-2}(k)$ for all $n \geq 5$. *(ii)* Explain why your formula is correct. [2 pts]

e. *(i)* Compute $P(k)$, the number of Vertex-colorings of a pyramid based on a pentagon. *(ii)* Explain why your formula is correct. [2 pts]

Solutions to 2011 Power Question

P-1. a. The answer is $\boxed{(3, 2, 3, 2)}$.

b. (i) The answer is that v_n equals 2 if n is even and 3 if n is odd. (ii) If n is even, then the polygon $P_1P_2 \dots P_n$ can be colored with two colors using one color of the vertices with odd subscripts and that other for vertices with even subscripts. If n is odd, two colors will not suffice because one of the colors would have to be used at least one more time than the other resulting in adjacent vertices with the same color. However, three colors will suffice. For example, for $i = 1, 2, \dots, n-1$ color the P_i with odd subscripts one color and those with even subscripts with a second color, and then color P_n with the third color.

c. (i) $\boxed{0 \text{ for all } n}$ (ii) A one-to-one correspondence between the colors of vertices and the colors of edges can be established by traversing the polygon counterclockwise, with each vertex the same color as the edge that follows it.

P-2. Let Face1 $= A_1B_1C_1D_1$ and Face2 $= A_2B_2C_2D_2$ be opposite faces of the cube with A_1 connected to A_2, B_1 connected to B_2, etc.

a. (i) $\boxed{2}$ (ii) For Face1, use one color for A_1 and C_1 and the other for B_1 and D_1. For the vertices on Face1, use the opposite color from the color used for the vertex on Face1 that is connected to it. Alternatively, consider the unit cube in the first octant of (x, y, z) with one vertex at the origin. Use one color for the vertices with coordinates whose sum is even, and the other for those with an odd sum.

b. (i) $\boxed{3}$ (ii) Since each vertex has three edges coming into it, the minimum number must be at least 3. That 3 colors suffice is shown by this argument: Face1 can be Edge-colored with two colors using one color for one pair of opposite edges and the other color for the other pair of edges. Face2 can similarly be colored with those same two colors. Use the third color for all of the edges joining Face1 and Face2.

c. (i) $\boxed{3}$ (ii) Since any pair of adjacent faces has a face adjacent to each, the minimum number must be at least 3. Three colors suffice because each of the three pairs of opposite faces can be colored using a different one of the three colors.

P-3. a. (i) The minimum number of colors is 3 if n is even and 4 if n is odd. (ii) By problem 1, for the vertices on the base of the pyramid, 2 colors are required if n is even and 3 if n is odd. Since B is connected to each of the vertices on the base, it must be a different color from all of the base vertices, thus increasing the minimum by 1.

b. (i) $\boxed{n}$ (ii) Since n edges come together at B, the minimum number must be n. Each edge on the base is adjacent to exactly 2 of the edges from B; and since those 2 edges use 2 colors and $n \geq 3$, there is an unused color for that can be used for that base edge.

c. (i) The minimum number of colors is 3 if n is even and 4 if n is odd. (ii) First consider the faces adjacent to B: If n is even, then these can be colored with 2 colors by alternating the colors for adjacent faces; and if n is odd, 2 colors will not suffice because one of the colors would have to be used at least one more time than the other resulting the same color the same color for two adjacent faces but 3 colors will suffice for these faces by, for example, for faces $A_iA_{i+1}B$ for $i = 1, 2, \ldots, n-1$ using one color when i is odd and a second color when i is even, and then coloring face A_nA_1B with the third color. The answer results since the color of the base must be one more than the color of any of the other faces.

P-4. a. (*i*) The minimum number of colors is 2 if n is even and 3 if n is odd. (*ii*) As in question **P**-1b, color the vertices of polygon $A_1A_2 \ldots A_n$ in 2 or 3 colors, depending on whether n is even or odd. Then, make the following correspondences of colors, $A_1 \leftrightarrow B_2$, $A_2 \leftrightarrow B_3$, $\ldots$, $A_{n-1} \leftrightarrow B_n$, $A_n \leftrightarrow B_1$. Since the colors on the "B-face" are just those on the "A-face" rotated by $(360/n)^\circ$, no $\overline{A_iB_i}$ edge will connect vertices of the same color.

b. (*i*) $\boxed{3}$ (*ii*) Since each vertex has 3 edges adjoining it, no number smaller than 3 will work. This shows that 3 colors suffice: If n is even, color the edges of the "A-face" with two colors as in question **P**-1c, then color the edges on the "B-face" identically to the "A-face", and finally color each $\overline{A_iB_i}$ face with the third color. If n is odd, color the edges of the "A-face" with three colors as in question **P**-1c, then color the edges of the "B-face" identically to the "A-face", and since there is the identical color combination at the $\overline{A_{i-1}A_i}\,\overline{A_ia_{i+1}}$ vertex on the A-face as there is at the $\overline{B_{i-1}B_i}\,\overline{B_iB_{i+1}}$ vertex on the B-face, for edge $\overline{A_iB_i}$ use the color not used on either $\overline{A_{i-1}A_i}$ or $\overline{A_iA_{i+1}}$.

c. (*i*) The minimum number of colors is 3 if n is even and 4 if n is odd. (*ii*) Any Edge-coloring of $A_1A_2 \ldots A_n$ corresponds to a Face-coloring of all the $A_iB_iB_{i+1}A_{i+1}$ faces, by just imagining the color of each $\overline{A_iA_{i+1}}$ edge 'bleeding' into the adjoining face and vice versa, so we need 2 colors for the $A_iB_iB_{i+1}A_{i+1}$ faces if n is even and 3 if n is odd. Since the two faces $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$ are adjacent to each of the $A_iB_iB_{i+1}A_{i+1}$ faces, one additional color is needed for these 'endcap' faces.

P-5. a. (*i*) $\boxed{k^3-3k^2+2k = k(k-1)(k-2)}$ (*ii*) $F_3(k) = k(k-1)(k-2)$ since each vertex must be a different color in a triangle.

b. (*i*) $\boxed{k^4-4k^3+6k^2-3k = k(k-1)(k^2-3k+3)}$ (*ii*) In a square $ABCD$, either A and C are the same color or A and C are different colors. When A and C are the same color, then there are k choices for that color and each of B and D can be one of $k-1$ colors, for a total of $k(k-1)^2 = k^3 - 2k^2 + k$ colorings. When A and C are different colors, then there are k choices for A, $k-1$ for C, and $k-2$ for each of B and D for a total of $k(k-1)(k-2)^2 = k^4 - 5k^3 + 8k^2 - 4k$ colorings. Therefore, the total number of colorings for the square is $(k^3 - 2k^2 + k) + (k^4 - 5k^3 + 8k^2 - 4k) = k^4 - 4k^3 + 6k^2 - 3k$ or $k(k-1)^2 + k(k-1)(k-2)^2 = k(k-1)[(k-1) + (k^2 - 4k + 4)]$.

Alternate Solution: Either 2 or 3 or 4 of the k colors can be used. If just 2 of the k colors are used, both opposite pairs of vertices of the square must be the same color, so there are k choices for the one pair and $k-1$ choices for the second pair for a total of $k(k-1)$ choices. If exactly 3 of the k colors are used, then one of the pairs of opposite vertices of the square must be the same color and there are two choices for the pair so there are $2k(k-1)(k-2)$ colorings in this case. If all 4 vertices are different colors, then there are $k(k-1)(k-2)(k-3)$ colorings for this case. Thus the total number of colorings is $k(k-1)+2k(k-1)(k-2)+k(k-1)(k-2)(k-3) = k(k-1)[1+(2k-4)+(k^2-5k+6)]$.

c. (*i*) $\boxed{k(k-1)(k-2)(k^2-2k+2)}$ (*ii*) In $ABCDE$ either (1) B and E are the same color or (2) B and E are different colors and B and D are different colors or (3) B and E are different colors and B and D are the same color. In case (1) there are k choices for $B = E$, $k-1$ choices for A, and $(k-1)(k-2)$ choices for C and D for a total $k(k-1)^2(k-2)$ colorings for this arrangement. In case (2) there are k choices for A, $k-1$ for B, $k-2$ for E, $k-2$ for D, and $k-2$ for C for a total of $k(k-1)(k-2)^3$ colorings for this arrangement. In case (3) there are k choices for A, $k-1$ choices for B, $k-2$ choices for $B = D$ and $k-1$ choices for c for a total of $k(k-1)^2(k-2)$ colorings for this arrangement. Totaling cases (1), (2), and (3) we have $2k(k-1)^2(k-2)+k(k-1)(k-2)^3 = k(k-1)(k-2)[2(k-1)+(k^2-4k+4)]$.

Alternate Solution: Either 3, 4, or 5 of the k colors can be used. If just 3 of the k colors are used, then one color is used for just one of the 5 vertices and the other non-adjacent vertices must be paired into vertices the same color, so there are $5k(k-1)(k-2)$ colorings with just 3 colors. If exactly 4 of the k colors are used, then some pair of vertices that could be joined by one of the 5 diagonals are the same color, so there are $5k(k-1)(k-2)(k-3)$ colorings in this case. If 5 of the k colors are used, then there are $k(k-1)(k-2)(k-3)(k-4)$ colorings in this case. Thus, there is a total of $k(k-1)(k-2)[5+(5k-15)+(k^2-7k+12)]$ colorings.

d. (*i*) $\boxed{(k-2, k-1)}$ (*ii*) Let $\mathcal{A} = A_1A_2A_3\dots A_n$ be any convex n-gon. Either (1) A_1 and A_3 are different colors, or (2) A_1 and A_3 are the same color.
Case 1: Diagonal $\overline{A_1A_3}$ could be inserted since A_1 and A_3 are different colors. The $(n-1)$-vertex polygon $\mathcal{A}_{\neq} = A_1A_3A_4\dots A_n$ can be Vertex-colored in $F_{n-1}(k)$ ways, and A_2 can be colored in any of the $k-2$ colors not used by A_1 or A_3. Thus, there are $(k-2)F_{n-1}(k)$ vertex-colorings of $\mathcal{A}$ with A_1 and A_3 different colors.
Case 2: Imagine coalescing vertices A_1 and A_3 into one vertex A_* since they are same color. There are $F_{n-2}(k)$ vertex-colorings of the $(n-2)$-vertex polygon $\mathcal{A}_= = A_*A_4A_5\dots A_n$. For each of these $F_{n-2}(k)$ vertex-colorings of $\mathcal{A}_=$, A_2 can be colored with one of the $k-1$ colors not used for A_1. Thus, there are $(k-1)F_{n-2}(k)$ vertex-colorings of $\mathcal{A}$ with A_1 and A_3 the same color.
Consequently, $F_n(k) = (k-2)F_{n-1}(k) + (k-1)F_{n-2}(k)$.
Note: A quick check to verify the algebra in parts **P**-5a, **P**-5b, and **P**-5c is to show that $(k-2)F_4(k) + (k-1)F_3(k) = F_5(k)$.

e. (*i*) $\boxed{k(k-1)(k-2)(k-3)(k^2-4k+5)}$ (*ii*) It is easiest to work from the alternative solution to **P**5-c: If 3 colors were used for the base pentagon, then a fourth color must be used for the vertex, so there are $(k-3)[5k(k-1)(k-2)]$ colorings for the pyramid in this case. If 4 colors were used for the base pentagon, then a fifth color must be used for the vertex, so there are $(k-4)[5k(k-1)(k-2)(k-3)]$ colorings for the pyramid in this case. If 5 colors were used for the base pentagon, then a sixth color must be used for the vertex, so there are $(k-5)[k(k-1)(k-2)(k-3)(k-4)]$ colorings for the pyramid in this case. Thus, the total number of colorings for the pyramid is

$$k(k-1)(k-2)(k-3)[5+(5k-20)+(k^2-9k+20) = k(k-1)(k-2)(k-3)(k^2-4k+5).$$

2011 Relay Problems

R1-1. The roots of $2x^2 - 20x + 48 = 0$ are $x = r$ and $x = h$ where $r > h$. The volume of a cylinder with height h and radius r is $V = n\pi$. Compute n.

R1-2. *Let N be the number you will receive.* The perimeter of a rectangle is 300, and its length is N. Compute the width of the rectangle.

R1-3. *Let N be the number you will receive.* Abe drives N miles per hour faster than Becky. They both leave at noon and drive 360 miles. Abe arrives at 6:00pm. Compute the number of minutes after Abe when Becky arrives.

R2-1. The line $2x + 3y = 72$ intersects the y-axis and x-axis at points A and B, respectively. The line $3x + 2y = 300$ intersects the x-axis and y-axis at points C and D, respectively. Compute the area of $ABCD$.

R2-2. *Let N be the number you will receive.* Compute $\lfloor \log_8(N) \rfloor$.

R2-3. *Let N be the number you will receive.* The perimeter of $\triangle ABC$ is $5N$, the lengths of all of its sides are positive integers, $AB = BC$, and AC is as small as possible. Compute the area of $\triangle ABC$.

2011 Relay Answers

R1-1. 144

R1-2. 6

R1-3. 40

R2-1. 7068

R2-2. 4

R2-3. $4\sqrt{5}$

2011 Relay Solutions

R1-1. Since the roots are $x = 4$ and $x = 6$, we have $r = 6$ and $h = 4$. Thus $V = \pi r^2 h = 144\pi$ and $n = \mathbf{144}$.

R1-2. Since the perimeter of an $L \times W$ rectangle is $P = 2L+2W$, $W = \frac{1}{2}(P-2L) = \frac{1}{2}(300-2\cdot 144) = \mathbf{6}$.

R1-3. Since Abe drove 360 miles in 6 hours, Abe drove at 60 miles per hour, so Becky drove at $60-N$ miles per hour. The number of hours it takes Becky to drive the 360 miles is $\frac{360}{60-N} = \frac{360-6N+6N}{60-N} = 6 + \frac{6N}{60-N}$. We convert the fractional part of an hour to minutes: $60 \cdot \frac{6N}{60-N} = \frac{360N}{60-N} = \frac{360\cdot 6}{60-6} = \frac{360\cdot 6}{9\cdot 6} = \mathbf{40}$.

R2-1. Note that $A = (0, 24)$, $B = (36, 0)$, $C = (100, 0)$, and $D = (0, 150)$. Let $O = (0, 0)$. Then both $\triangle AOB$ and $\triangle COD$ are right triangles, and the area of $ABCD$ is the difference between their areas; namely, $\frac{1}{2}(100)(150) - \frac{1}{2}(36)(24) = \mathbf{7068}$.

R2-2. Since $\lfloor \log_8(N) \rfloor = k$ if and only if $8^k \leq N < 8^{k+1}$, while awaiting the arrival of N we compute $8^2 = 64$, $8^3 = 512$, $8^4 = 4096$, $8^5 = 32768$, $\ldots$. Since $8^4 < 7068 < 8^5$, it follows that $\lfloor \log_8(N) \rfloor = \mathbf{4}$.

R2-3. If $5N$ is odd, let $5N = 2k+1$, and then $AB = BC = k$ and $AC = 1$ so the area will be $\frac{1}{2}(1)\sqrt{k^2 - \left(\frac{1}{2}\right)^2} = \frac{1}{4}\sqrt{4k^2-1}$. If $5N$ is even, let $5N = 2k+2$, and then $AB = BC = k$ and $AC = 2$ so the area will be $\frac{1}{2}(2)\sqrt{(k^2 - 1^2)} = \sqrt{k^2-1}$. Since $5N = 20 = 2(9) + 2$, the area is $\sqrt{9^2 - 1} = \mathbf{4\sqrt{5}}$.

2011 Tiebreaker Problems

TB-1. In $\triangle ABC$, $\angle C$ is a right angle, $AC = 6$, and $BC = 8$. Point D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\overline{AB}$. Circles C_1 and C_2 are drawn such that C_1 is tangent to all three sides of $\triangle BDC$ and C_2 is tangent to all three sides of $\triangle ADC$. Compute the sum of the areas of C_1 and C_2.

TB-2. Let $\lfloor \log_{11} n \rfloor$ be the largest integer less than or equal to $\log_{11} n$. Compute

$$\sum_{n=1}^{2011} \lfloor \log_{11} n \rfloor.$$

2011 Tiebreaker Answers

TB-1. $\dfrac{145\pi}{36}$

TB-2. 4573

2011 Tiebreaker Solutions

TB-1. The area of a triangle can be expressed as $A = rs$, where r is the radius of the incircle and s is the semiperimeter. So, for $\triangle BDC$, it follows that $12 = 9r_1$, so $r_1 = \frac{4}{3}$. Similarly, for $\triangle ADC$, it follows that $12 = 8r_2$, so $r_2 = \frac{3}{2}$. The desired area is $\pi\left(\left(\frac{4}{3}\right)^2 + \left(\frac{3}{2}\right)^2\right) = \mathbf{\frac{145\pi}{36}}$.

TB-2. For $1 \le n \le 10$, $\lfloor \log_{11} n \rfloor = 0$.
For $11 \le n \le 120$, $\lfloor \log_{11} n \rfloor = 1$.
For $121 \le n \le 1330$, $\lfloor \log_{11} n \rfloor = 2$.
For $1331 \le n \le 2011$, $\lfloor \log_{11} n \rfloor = 3$.
Therefore, $\sum_{n=1}^{2011} \lfloor \log_{11} n \rfloor = 110 \cdot 1 + 1210 \cdot 2 + 681 \cdot 3 = \mathbf{4573}$.

2012 Contest at Susquehanna Valley High School (Southern Tier)

Contents

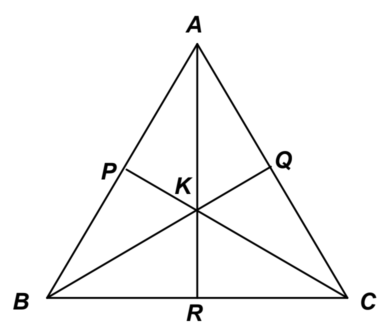
2012 Team Problems

T-1. Xavier starts walking down a straight road at 3:00 PM, and walks at 3 miles per hour. Yolanda starts walking down the same straight road at 4:00 PM, and she walks at 4 miles per hour. Zachary begins walking down the same straight road at 5 miles per hour. If the three walkers reach the same point at the same time, compute the time (PM) at which Zachary should begin walking.

T-2. Call a positive integer *"Two-Prime-Linked"*, or TPL for short, if (a) all the digits are different and (b) each pair of adjacent digits forms a two-digit prime. For example, 417 is a TPL integer since $4, 1, 7$ are all different and 41 and 17 are two-digit primes. Compute the smallest five-digit TPL integer.

T-3. Compute the least positive integer n for which n and $n+1$ are both the product of three distinct prime factors.

T-4. ABC is an isosceles triangle with base $\overline{BC}$. P, Q, and R are midpoints of the sides that contain them. K is the intersection of the medians, as shown. If $AR = BC = 12$, compute $AC + KC$.



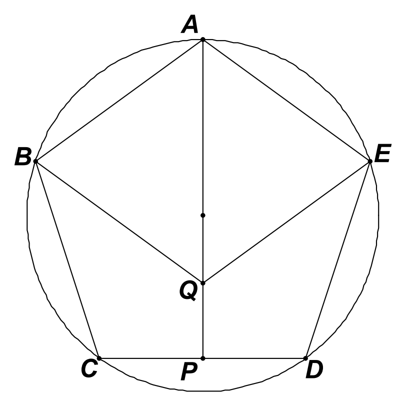
T-5. Consider three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Let $a_{n+1} = \lceil\sqrt{b_n c_n}\rceil$, $b_{n+1} = \lceil\sqrt{a_n c_n}\rceil$, and $c_{n+1} = \lceil\sqrt{a_n b_n}\rceil$, where $\lceil x\rceil$ is the least integer equal to or greater than x. Given that $a_1 = 2$, $b_1 = 8$, and $c_1 = 32$, compute a_{10}.

T-6. Suppose $x^2 + y^2 = 68$ and $\log_2 x + \log_2 y = 4$. Compute $|x + y|$.

T-7. Dick's stamp collection contains commemorative (C), definitive (D) and back-of-the-book (B) stamps. In 2011, his collection contained 4242 face different U.S. stamps. At least two-thirds of the stamps were commemorative and the ratio of the definitive to the back-of-the-book was $4 : 1$. Since then he has added 6 definitive and 42 back-of-the-book stamps and many more commemorative stamps to his collection and the $C : D : B$ ratio is now exactly $20 : 7 : 2$. Compute the number of stamps in his collection.

T-8. Call an integer a 012-*integer* if it can be written using only the digits 0, 1, and 2, and it uses each of these digits at least once. Compute the number of 012-integers in $\{1, 2, \cdots, 999999\}$. Examples: The smallest 012-integer is 102, the largest in this set is 222210, and this year, 2012, is another example of a 012-integer.

T-9. Let $ABCDE$ be a regular pentagon inscribed in a circle of radius R. P is the midpoint of $\overline{CD}$. Q is the reflection of A across $\overline{BE}$. Compute PQ in terms of R.



T-10. The faces of a polyhedron $\mathcal{P}$ consist of 8 equilateral triangles and 6 regular octagons. The volume of $\mathcal{P}$ is $189 + 126\sqrt{2}$. Compute the total surface area of $\mathcal{P}$.

2012 Team Answers

T-1. 4:36 PM

T-2. 23179

T-3. 230

T-4. $6\sqrt{5}+2\sqrt{13}$

T-5. 10

T-6. 10

T-7. 4698

T-8. 488

T-9. $R\left(\frac{3\sqrt{5}-5}{4}\right)$

T-10. $108+108\sqrt{2}+18\sqrt{3}$

2012 Team Solutions

T-1. We take advantage of the fact that distance is the product of rate and time. We need Xavier's distance to equal Yolanda's distance, so $3(t+1) = 4t$, which means the trip takes Yolanda 3 hours and the distance is $3 \cdot 4 = 12$ miles. If the trip takes Yolanda 3 hours, the trio meet at 7:00 PM. Thus, Zachary will take the trip in $\frac{12}{5} = 2.4$ hours, which is 2 hours and 24 minutes. He therefore has to begin at **4:36 PM**.

T-2. Every two-digit prime ends in 1, 3, 7, or 9, so all but the first digit must be chosen from these four. Listing these end-digits in increasing order 1379 yields the two-digit primes 13, 37, and 79. However, since 21 is not prime, 41379 is the smallest five-digit TPL integer attainable with the end-digits in increasing order. Listing the end-digits in the order 3179 for the next-smallest value of the last four digits yields the two-digit primes 31, 17, and 79. Furthermore 23 is also a two-digit prime. Thus, **23179** is the smallest five-digit TPL integer.

T-3. Because any multiple of 4 would not have distinct prime factors, either n or $n+1$ must be congruent to 2 (modulo 4). With that in mind, consider the odd number abc and check the even neighbor that isn't a multiple of 4. So, $3 \cdot 5 \cdot 7 = 105$, but $106 = 2 \cdot 53$; and $3 \cdot 5 \cdot 11 = 165$, but $166 = 2 \cdot 83$; and $3 \cdot 5 \cdot 13 = 195$, but $194 = 2 \cdot 97$. However, $3 \cdot 7 \cdot 11 = 231$ and $230 = 2 \cdot 5 \cdot 23$. Thus, our answer is **230**. (Just to be sure, verify that $3 \cdot 5 \cdot 17 > 231$.)

T-4. In right $\triangle ARC$, $AC^2 = 12^2 + 6^2 = 180$, so $AC = 6\sqrt{5}$. Since medians intersect at a common point (the centroid) that divides each median into a $2:1$ ratio, $KR = 4$. In right $\triangle KRC$, $KC^2 = 4^2 + 6^2 = 52$, so $KC = 2\sqrt{13}$. Thus, $AC + KC = \mathbf{6\sqrt{5} + 2\sqrt{13}}$.

T-5. Keep in mind the symmetry of the elements for any value of n. Also keep in mind the following: $\lceil \sqrt{x(x+1)} \rceil = x+1$ and $\lceil \sqrt{x(x+2)} \rceil = x+1$. Let D_n represent the ordered triple (a_n, b_n, c_n). So, $D_1 = (2, 8, 32)$, $D_2 = (16, 8, 4)$, $D_3 = (6, 8, 12)$, $D_4 = (10, 9, 7)$, and $D_5 = (8, 9, 10)$. From here, all values quickly converge to **10**.

T-6. The second equation can be rewritten as $\frac{\log x}{\log 2} + \frac{\log y}{\log 2} = 4$. This means that $\log x + \log y = 4 \log 2 = \log 16$. Using the Product Property of Logs, we know that $\log(x \cdot y) = \log 16$, and thus $xy = 16$. Now, combine that with the first equation in the following way. We know that $x^2 + 2xy + y^2 = 68 + 2 \cdot 16 = 100$, so $(x+y)^2 = 100$, and $|x+y| = \sqrt{100} = \mathbf{10}$.

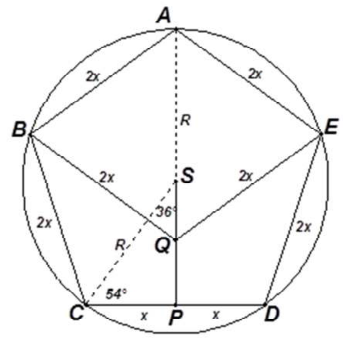
T-7. At least 2828 stamps are commemorative, leaving 1414 to be divided into two parts in a $4:1$ ratio. 1414 is not divisible by 5. Thus, more stamps must have been commemorative. Some possibilities are: $2832 \to 1410(1128, 282)$, $2837 \to 1405(1124, 281)$, $2842 \to 1400(1120, 280)$. In general, $(D, B) = (1128 - 4t, 282 - t)$. Now $\frac{1128-4t+6}{282-t+42} = \frac{7}{2} \to \frac{1134-4t}{324-t} = \frac{7}{2}$, so $2268 - 8t = 2268 - 7t$, and thus $t = 0$. Thus, in 2011, $(C, D, B) = (2832, 1128, 282)$. Currently, $\frac{2832+x}{1134} = \frac{20}{7} \to 7x = 20 \cdot 1134 - 7 \cdot 2832$, so $x = 20 \cdot 162 - 2832 = 3240 - 2832 = 408$. Thus, the collection now contains $4242 + 408 + 6 + 42 = \mathbf{4698}$ stamps.

T-8. Let $f(n)$ be the number of n-digit 012-integers. Note that $f(1) = f(2) = 0$, so we need to compute $f(3) + f(4) + f(5) + f(6)$. If x is an n-digit 012-integer, then the most significant digit of x is 1 or 2 and remaining $n-1$ digits of x form a sequence of $n-1$ of the three digits $\{0, 1, 2\}$ which may contain the most significant digit and must contain at least one of each of the other two digits. By the Inclusion-Exclusion Principle, there are $3^{n-1} - 2 \cdot 2^{n-1} + 1$ choices for these remaining $n-1$ digits. Therefore, $f(n) = 2(3^{n-1} - 2^n + 1)$. Consequently, $f(3)+f(4)+f(5)+f(6) = 2[(3^2-2^3+1)+(3^3-2^4+1)+(3^4-2^5+1)+(3^5-2^6+1)$, or $2[(3^2+3^3+3^4+3^5) - (2^3+2^4+2^5+2^6) + (4)] = 2\left[\frac{3^6-3^2}{3-1} - \frac{2^7-2^3}{2-1} + 4\right] = \mathbf{488}$.

Alternate Solution: First note that $f(3) = 4$ because of $\{102, 120, 201, 210\}$. For $n > 3$, let x be an n-digit 012-integer. Either x's leftmost $n-1$ digits form a 012-integer and there are three choices for the units digit, or x's most significant digit is one of two choices and the next $n-2$ digits form a non-constant sequence with a range of two digits (the most significant digit and one of the other two choices) and the units digit is the digit not used in the sequence. Consequently, $f(n) = 3f(n-1) + 2 \cdot 2 \cdot (2^{n-2} - 1) \cdot 1 = 3f(n-1) + 4(2^{n-2} - 1)$. Thus $f(4) = 3(4) + 4(3) = 24$, and $f(5) = 3(24) + 4(7) = 100$, and $f(6) = 3(100) + 4(15) = 360$, so $f(3) + f(4) + f(5) + f(6) = 4 + 24 + 100 + 360 = 488$.

This problem was originally written by Dr. Leo J. Schneider, who wrote NYSML contests from 2001 until his death in 2010. We include this problem to honor his memory.

T-9. Let $(2x)$ denote the sides of $ABCDE$. Let S denote the center of the circumscribed circle. In $\triangle PCS$, $SP = R\cos 36°$ and $x = R\sin 36°$. $AP = AS + SP = R + R\cos 36° = R(1 + \cos 36°)$. Since 36° is a special angle, we know $\cos 36° = \frac{\sqrt{5}+1}{4}$. Substituting, $AP = R\left(1 + \frac{\sqrt{5}+1}{4}\right) = R\left(\frac{\sqrt{5}+5}{4}\right)$. In isosceles $\triangle ABQ$, $m\angle BAQ = 54°$ which forces $m\angle ABQ = 72°$. Dropping a perpendicular from B to M, the midpoint of $\overline{AQ}$, creates a $36° - 54° - 90°$ right $\triangle ABM$. Thus, we have $AQ = 4x\sin 36° = 4R\sin^2 36°$. Knowing $\cos 36°$, we can determine an exact value for $\sin^2 36°$: $\sin^2 36° = 1 - \cos^2 36°$, which is $1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{16-(5+2\sqrt{5}+1)}{16} = \frac{10-2\sqrt{5}}{16}$. Finally, $PQ = AP - AQ = R\left(\frac{\sqrt{5}+5}{4}\right) - 4R\left(\frac{10-2\sqrt{5}}{16}\right)$, which simplifies to $R\left(\frac{\sqrt{5}+5-(10-2\sqrt{5})}{4}\right) = \mathbf{R\left(\frac{3\sqrt{5}-5}{4}\right)}$.



T-10. The number of vertices of $\mathcal{P}$ is $V = E - F + 2 = \frac{8\cdot 3+6\cdot 8}{2} - 14 + 2 = 24$ by Euler's Formula. Since each interior angle of each octagon measures $135°$ and $3 \cdot 135 > 360$, it follows that the corners of at most two octagons come together at each vertex of $\mathcal{P}$. But, since there are $6 \cdot 8 = 48$ corners on the octagons and only 24 vertices, we must have two corners of octagons come together at each vertex of $\mathcal{P}$. Furthermore, since there must be at least three corners of polygons at each of the 24 vertices of $\mathcal{P}$ and the total number of corners on the triangles is $3 \cdot 8 = 24$, each vertex of $\mathcal{P}$ is formed by the corners of two octagons and one equilateral triangle. It follows that, the faces adjacent to the octagon and around the edges of the octagon must alternate between the triangles and octagons. Thus, $\mathcal{P}$ can be formed by starting with a cube and removing, at each vertex of the cube, a tetrahedron with an equilateral triangle as a base and three isosceles right triangles coming together its vertex. The length s of the side of each equilateral triangle must equal the remaining length of the part of each edge of the cube that remains after the tetrahedrons are removed, thus forming the regular octagons. Each octagonal face of $\mathcal{P}$ consists of an $s \times \left(s + s\sqrt{2}\right)$ rectangle and two trapezoids with altitudes $s\sqrt{2}$ and bases of lengths s and $s + s\sqrt{2}$. Therefore the total surface area of $\mathcal{P}$ is

$$6\left(s\left(s+s\sqrt{2}\right)+2\frac{s\sqrt{2}}{2}\left(2s+\frac{s\sqrt{2}}{2}\right)\right)+8\left(\frac{s^2\sqrt{3}}{4}\right)=\left(12+12\sqrt{2}+2\sqrt{3}\right)s^2.$$

Let h be the length of the altitude of each tetrahedron. Then $h^2 + \left(\frac{2}{3}\left(\frac{s\sqrt{3}}{2}\right)\right)^2 = \left(\frac{2s}{\sqrt{2}}\right)^2$ so $h = \sqrt{\frac{s}{6}}$. The edge-length of the cube was $s + s\sqrt{2}$. The volume of $\mathcal{P}$ is

$$\left(s+s\sqrt{2}\right)^3 - 8\left(\frac{1}{3}\left(\frac{s^2\sqrt{3}}{4}\right)\left(\frac{s}{\sqrt{6}}\right)\right) = \left(7+\frac{14\sqrt{2}}{3}\right)s^3 = 189+126\sqrt{2}$$

so $s = 3$ and the total surface area of $\mathcal{P}$ is $9\left(12+12\sqrt{2}+2\sqrt{3}\right) = \mathbf{108+108\sqrt{2}+18\sqrt{3}}$.

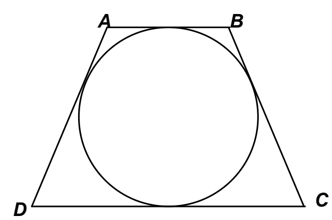
2012 Individual Problems

I-1. For all real values of x, compute the minimum value of $|x+2|+|x-5|$.

I-2. For some integer $x > 13$, the sum of the first and third (lower and upper) quartiles scores, the median, and the mean of the set of numbers $\{1, 1, 2, 3, 5, 8, 13, x\}$ is 34. Compute x.

I-3. Given a sequence $\{a_n\}$ where $a_0 = 2$, $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, and for all $n \geq 4$, $a_n = 2a_{n-1} - 0a_{n-2} + 1a_{n-3} - 2a_{n-4}$. Compute a_{2012}.

I-4. Trapezoid $ABCD$ is circumscribed about a circle. $\overline{AB} \parallel \overline{CD}$, $AD = BC = 40$. The radius of the circle is 15. Compute the area of $ABCD$.



I-5. Let $\binom{n}{r}$ denote the number of ways to choose r objects from a group of n objects. Given that $\dfrac{\binom{n}{3}}{\binom{n}{2}} = 10$, compute n.

I-6. Triangle JET has vertices at $J(2, 10)$, $E(2, 2)$, and $T(8, 2)$. The circumcircle of $\triangle JET$ passes through each of the vertices of $\triangle JET$. The incircle of $\triangle JET$ is internally tangent to each side of $\triangle JET$. Compute the area in between the circumcircle and the incircle.

I-7. Compute $\log_{\left(8^{6^4}\right)}\left(4^{6^8}\right)$. Recall that $a^{b^c} = a^{(b^c)}$.

I-8. The graph of $(x^2+y^2-2(3x+4y))(y+2-|x-2|) = 0$ partitions the plane into 6 regions whose areas are $A_1, A_2, \cdots, A_6$ named so $0 < A_1 < A_2 < A_3 < A_4 < A_5 = A_6 = \infty$. Compute A_3.

I-9. In hexagon $ABCDEF$, 5 of the angles each measure $135°$ and 4 of the sides each measure $\sqrt{2}$. Compute the area of $ABCDEF$.

I-10. Compute the number of distinct positive integer factors of $2^{16} - 1$.

2012 Individual Answers

I-1. 7

I-2. 111

I-3. 1

I-4. 1200

I-5. 32

I-6. 21π

I-7. 864

I-8. $\dfrac{25\pi}{4} - \dfrac{25}{2}$

I-9. $6 + 5\sqrt{2}$

I-10. 16

2012 Individual Solutions

I-1. Consider three points A, B, and C on the number line with coordinates -2, 5, and x respectively. The given expression represents the sum of the distances from x to -2 and x to 5. If C is between A and B (or $C = A$ or $C = B$), then $AC + CB = AB = 5 - (-2) = 7$. If C is to the left of -2, then $CA + AB = CA + 7 > 7$. If C is to the right of 5, then $CB + AB = CB + 7 > 7$. Our minimum is **7**.

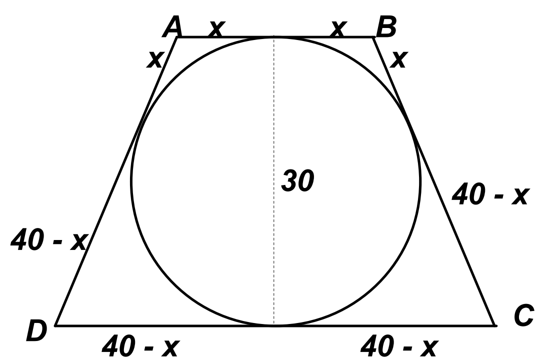
I-2. Let μ be the mean, Q_1 and Q_3 be the first and third quartile scores, and M be the median. Then

$$34 = Q_1 + M + Q_3 + \mu = \frac{1+2}{2} + \frac{3+5}{2} + \frac{8+13}{2} + \frac{1+1+2+3+5+8+13+x}{8} = \frac{161+x}{8}$$

so $161 + x = 272$ and $x = \mathbf{111}$.

I-3. We see that $a_4 = 2(2) - 0(1) + 1(0) - 2(2) = 0$, $a_5 = 2(0) - 0(2) + 1(1) - 2(0) = 1$, $a_6 = 2(1) - 0(0) + 1(2) - 2(1) = 2$, $a_7 = 2(2) - 0(1) + 1(0) - 2(2) = a_4 = 0$, and so on. By extension, $a_{k+3} = a_k$, $k \geq 0$, and $a_{2012} = a_{2009} = \cdots = a_3 = \mathbf{1}$.

I-4. Consider the diagram.



Notice that $AB + CD = 2x + 2(40 - x) = 2x + 80 - 2x = 80$. So the area of $ABCD$ is $\frac{1}{2}h(AB + CD) = \frac{1}{2} \cdot 30 \cdot 80 = \mathbf{1200}$.

I-5. The equation may be rewritten as $\dfrac{\dfrac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1}}{\dfrac{n(n-1)}{2 \cdot 1}} = \frac{n-2}{3} = 10$, so $n = \mathbf{32}$.

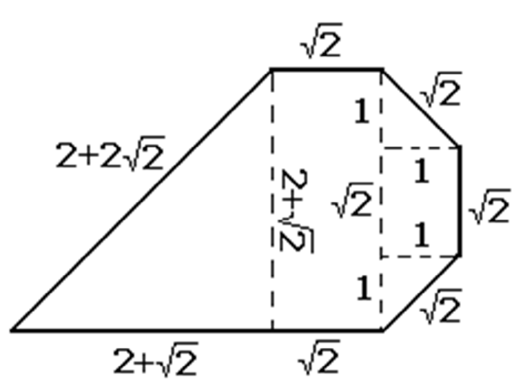
I-6. Since $\triangle JET$ is a right triangle, its circumcenter is at the midpoint of the hypotenuse (think: it's where the perpendicular bisectors of the sides come together). So the circumcircle is centered at $(5,6)$ and the circumradius is $\sqrt{(5-2)^2+(6-10)^2}=5$. The inradius can be found without knowing where the incenter is, with the help of the formula for the area of a triangle $A=r\cdot s$, where r is the inradius and s is the semiperimeter of the triangle. We know the area of the triangle is $\frac{1}{2}\cdot 6\cdot 8=24$, so we solve $24=r\cdot 12$ to obtain $r=2$. The area between the two circles is $5^2\cdot\pi-2^2\cdot\pi=\mathbf{21\pi}$.

I-7. Recall that exponentiation is right-associative. $\log_{\left(8^{6^4}\right)}\left(4^{6^8}\right)=\dfrac{\log_n\left(4^{6^8}\right)}{\log_n\left(8^{6^4}\right)}=\dfrac{6^8\log_n 4}{6^4\log_n 8}$. This simplifies to $6^4\log_8 4=1296\cdot\frac{2}{3}=\mathbf{864}$.

I-8. Since a product is zero if and only if one of its factors is zero, the curves parts of which bound the regions are the circle $(x-3)^2+(y-4)^2=25$ with center $(3,4)$ and radius 5 and the 'big $\boldsymbol{V}$' $y=|x-2|-2$ with vertex at $(2,-2)$ and the rays $y=x-4$ when $x\geq 2$ and $y=-x$ when $x\leq 2$ as arms. A quick sketch shows that the region with area A_3 is bounded by a segment of $y=x-4$ and the circle. Solving $(x-3)^2+(y-4)^2=25$ and $y=x-4$ simultaneously yields the intersection points $(3,-1)$ and $(8,4)$. Thus A_3 equals $1/4$ the area of a circle of radius 5 minus the area of an isosceles right triangle, so $A_3=\frac{1}{4}\pi 5^2-\frac{1}{2}5^2=\boldsymbol{\frac{25\pi}{4}-\frac{25}{2}}$.

This problem was originally written by Dr. Leo J. Schneider, who wrote NYSML contests from 2001 until his death in 2010. We include this problem to honor his memory.

I-9. The sixth angle must measure $4(180)-5(135)=45°$. The sides of that angle must be longer than the other 4 sides in order to accommodate the obtuse angles, so those other 4 sides must each measure $\sqrt{2}$. This gives the following diagram (with partitions into simple areas).

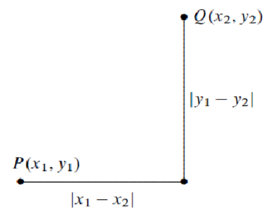


Our area is $\frac{1}{2}(2+\sqrt{2})^2+\sqrt{2}\cdot(2+\sqrt{2})+2\cdot\frac{1}{2}\cdot 1^2+1\cdot\sqrt{2}$, or $\mathbf{6+5\sqrt{2}}$.

I-10. We factor $2^{16}-1$ as a difference of two squares: $(2^8-1)(2^8+1)$. Since the second factor is one more than 2 to a power of 2, it is prime (look it up!). So we continue to factor the other factor. $2^{16}-1=(2^8-1)(2^8+1)=(2^4-1)(2^4+1)(2^8+1)=(2^2-1)(2^2+1)(2^4+1)(2^8+1)$. Since each of these four primes may or may not be in a particular factor of $2^{16}-1$, the number of factors is therefore $2\cdot 2\cdot 2\cdot 2=\mathbf{16}$.

Power Question 2012: Taxicab Geometry

In the ordinary Cartesian plane, $\Re^2$, the Euclidean distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$, written $d(P, Q)$, is $d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
In Taxicab geometry, the axes and coordinates are the same as in $\Re^2$, but Taxi distance is measured differently. In Taxicab geometry $d(P, Q)$ is defined $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.



It is defined this way because in a city with only north-south and east-west roads, it gives the distance a cab would have to drive to get from P to Q, as indicated in the diagram. We will refer to distance in $\Re^2$ as Euclidean and write $d_E(P, Q)$. Taxi distance will be written as $d_T(P, Q)$.

A circle with center C and radius $r > 0$ in Euclidean geometry is defined as the set of points P in the plane whose distance from C is r. If $C = (x_0, y_0)$ and $P = (x, y)$, the equation of the circle can be expressed as $d_E(P, C) = r$, or $\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$, which is usually written as $(x - x_0)^2 + (y - y_0)^2 = r^2$. We define a *Taxi circle with center C and radius r* as the set of points P in the plane such that $d_T(P, C) = r$, leading to the equation $|x - x_0| + |y - y_0| = r$. For notational purposes we will write $C_E(p, q, r)$ for the Euclidean circle with center (p, q) and radius r, and $C_T(p, q, r)$ for the Taxi circle with the same center and radius.

P-1. a. Sketch the Taxi circle with center $(0, 0)$ and radius 3. [3 pts]

b. Write the equation of the Taxi circle with center $(3, 3)$ and radius 5, and determine its perimeter. [3 pts]

P-2. Given three distinct collinear points in the plane, there is no Euclidean circle passing through all three. Consider the collinear points $A(0, 6)$, $B(3, 3)$, and $C(6, 0)$. Show by means of a diagram that there is an infinite number of Taxi circles containing A, B, and C. Then show algebraically or analytically that there is an infinite number of Taxi circles containing A, B, and C. [6 pts]

P-3. Let P, Q, and R be three distinct points on a non-vertical line. Suppose the slope of the line is not 1 or -1. Prove there is no Taxi circle passing through all three points. [5 pts]

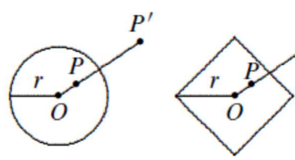
P-4. a. In Taxicab geometry, draw a graph of the set of points equidistant from $O(0, 0)$ and $N(6, 4)$. [3 pts]

b. There is a Taxi circle $C_T(p, q, r)$ passing through the points $X(1, 2)$, $Y(6, 5)$, and $Z(5, -2)$. Compute the ordered triple (p, q, r). [3 pts]

P-5. Given any three non-collinear points, there is exactly one Euclidean circle passing through all three points. Give an example, with proof, of three non-collinear points in the plane for which there is **no** Taxi circle passing through all three of them. [4 pts]

P-6. Given any three non-collinear points, there is exactly one Euclidean circle passing through all three points. Give an example, with proof, of three non-collinear points in the plane for which there is **an infinite number** of Taxi circle passing through all three of them. Your proof may be diagrammatic or algebraic. [4 pts]

Let $C_E(O, r)$ be a Euclidean circle. The inversion of the point P in the circle C is the point P' on the ray $\overrightarrow{OP}$ such that $d_E(O, P) \times d_E(O, P') = r^2$. Inversion in $C_T(O, r)$ is defined similarly, but so that $d_T(O, P) \times d_T(O, P') = r^2$. See the accompanying diagrams.



The rest of the questions will involve inverting in the unit Taxi circle, $C_T(0, 0, 1)$. Note that the origin is not in the domain or range of the inversion.

P-7. Compute the coordinates of P' under inversion in the unit Taxi circle, if P is the point with coordinates $(0.1, 0.2)$. [3 pts]

P-8. Under inversion in the unit circle, notice that since P and P' are on the same ray through the origin, for $P(x, y)$, $P' = (\lambda x, \lambda y)$ for some $\lambda > 0$. Show that $\lambda = \frac{1}{(|x|+|y|)^2}$. That is, under inversion in the unit circle, $P(x, y) \to P'\left(\frac{x}{(|x|+|y|)^2}, \frac{y}{(|x|+|y|)^2}\right)$. [4 pts]

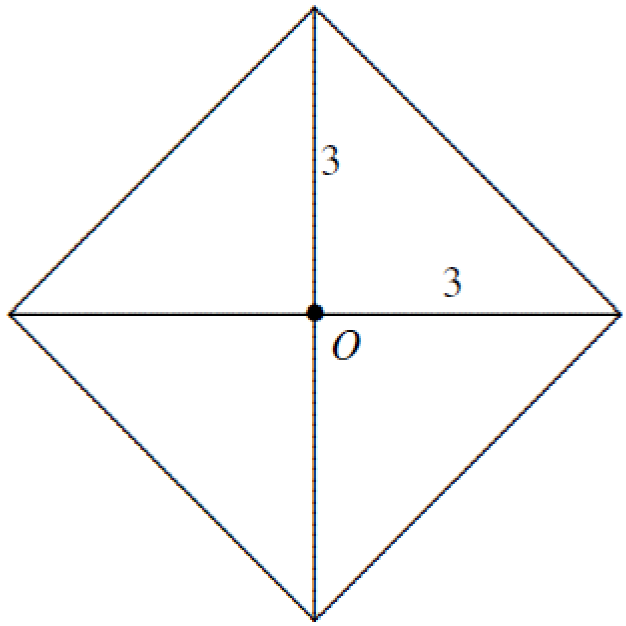
P-9. Show that under inversion in the unit Taxi circle, the image of $C_T(0, 0, r)$ is $C_T(0, 0, \frac{1}{r})$ for all $r > 0$. [4 pts]

P-10. Show that for any point P, under inversion in the unit Taxi circle, $P'' = P$. [3 pts]

P-11. Consider the Euclidean parabola $y = x^2$, $x > 0$. The curve may be expressed parametrically as $P(t) = (t, t^2)$, $t > 0$. Prove that when the parabola is inverted in the unit Taxi circle, there are points P' that are arbitrarily close to the origin and points P' that are arbitrarily far from the origin. [5 pts]

Solutions to 2012 Power Question

P-1. a. By definition, we know $|\alpha| = \alpha$ if $\alpha \geq 0$ and $|\alpha| = -\alpha$ if $\alpha < 0$. So, for points in the first quadrant, $|x| + |y| = 3$ becomes $x + y = 3$, and there are similar results in the other quadrants. The resulting picture looks like the diagram below.



A Taxi circle is thus a Euclidean square whose sides have slopes ± 1. Henceforth, we will speak of the sides of the circle to refer to the sides of the Euclidean square.

b. The equation of the Taxi circle centered at (h, k) of Taxi radius r is $|x - h| + |y - k| = r$, so our answer is $\mathbf{|x - 3| + |y - 3| = 5}$. Its perimeter is the sum of the lengths of four sides, each of length $5\sqrt{2}$, so the perimeter is $\mathbf{20\sqrt{2}}$.

P-2. The diagram can take many forms. For example, take $(6, 0)$ as the bottom corner and extend the segment through $(6, 0)$ and $(0, 6)$ as far as desired. Use this segment to construct the square. This can be done in infinitely many ways.

P-3. Note that if any two of the points were on the same side of the circle, the slope between them would be ± 1, contrary to hypothesis. Therefore, the points must be on three sides of the circle. But no line can intersect three sides of a square, and we have a proof by contradiction.

P-4. a. The set of points is the union of three sets of points: the lattice points on the line segment connecting $(1, 4)$ and $(5, 0)$, the lattice points on the ray $x = 1$ above $(1, 4)$, and the lattice points on the ray $x = 5$ below $(5, 0)$.

b. $\boxed{\mathbf{(5, 4, 4)}}$ Using $(1, 2)$ and $(5, -2)$ as "corners" draw the line through $(6, 5)$ with slope -1. The square formed has $(4, 6)$ and $(8, 2)$ as the other corners. The center of the square is $(5, 4)$ and the radius of the Taxi circle is 4.

P-5. Consider the points (for example) $(0, 1)$, $(2, 2)$, and $(2, 0)$. The three points do not lie on any Taxi circle. Every point on a Taxi circle lies on a segment with slope ± 1. Drawing a sketch of the six lines with those slopes that pass through the three points, one can see that no Taxi circle can be constructed through the three points.

P-6. Answers will vary, as will the explanations.

P-7. $\boxed{(\mathbf{1/9}, \mathbf{2/9})}$ Since P' lies on the ray through the origin passing through $(0.1, 0.2)$, it must have coordinates $(0.1\lambda, 0.2\lambda)$ for some λ. Then $OP = 0.1 + 0.2 = 0.3$ and $OP' = 0.3\lambda$. $OP \times OP' = 1^2 = 1 \to \lambda = 1/0.9$, so P' has coordinates $(\mathbf{1/9}, \mathbf{2/9})$.

P-8. As in the analysis for Question **P-7**, $OP = |x|+|y|$, and $OP' = \lambda(|x|+|y|)$. Then $OP \times OP' = 1 \to \lambda = \frac{1}{(|x|+|y|)^2}$.

P-9. Let $P(X, Y)$ be on $C_T(0, 0, r)$. Then $|x| + |y| = r$, so P' has coordinates $(x/r^2, y/r^2)$, so $OP' = r/r^2 = 1/r$, so P' is on the circle $C_T(0, 0, r)$.

P-10. This follows from the fact that $\frac{1}{1/r} = r$. We leave the details to the reader to fill in.

P-11. We have $P(t, t^2) \to OP = t + t^2$, and $OP' = \frac{1}{t+t^2}$. By choosing small t, $t + t^2$ can be made as small as desired, so that $OP' = \frac{1}{t+t^2}$ can be made as large as desired. Thus, there are points P' arbitrarily far from the origin. By choosing large t, $t + t^2$ can be made as large as desired, so that $OP' = \frac{1}{t+t^2}$ can be made as small as desired. Thus, there are points P' arbitrarily close to the origin. More formally, let $\epsilon > 0$ be chosen. Since ϵ is supposed to be small, assume $\epsilon < 2$, so that $\epsilon/2 < 1$ and $(\epsilon/2)^2 < \epsilon/2$. Thus, if $t < \epsilon/2$, $t^2 + t = (\epsilon/2)^2 + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$. This shows that $t + t^2$ can be made smaller than any desired quantity ϵ. Now let M be chosen. If $t > \sqrt{M}$, $t^2 + t > M + \sqrt{M} > M$. This shows that $t^2 + t$ can be made as large as desired, which implies $\frac{1}{t^2+t}$ can be made as small as desired.

2012 Relay Problems

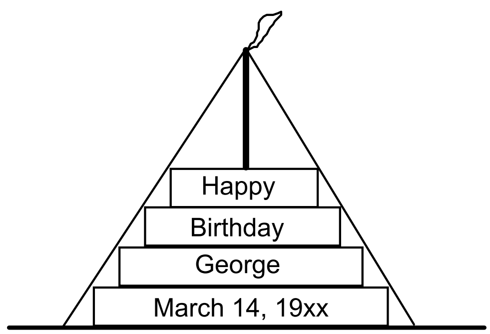
R1-1. Compute the sum: $1+2+3+4+6+8+7+10+13+10+14+18+13+18+23+16+22+28$.

R1-2. *Let N be the number you will receive.* In a garden are three-leaf clovers and four-leaf clovers. There are 16 fewer four-leaf clovers than three-leaf clovers. The total number of leaves is N. Compute the number of four-leaf clovers in the garden.

R1-3. *Let N be the number you will receive.* Parallelogram $MATH$ has vertices $M(2,4)$ and $A(5,8)$. T could be at any point on the line $y=N$ except the point (k,N). Compute k.

R2-1. Pi Day was last celebrated on 03/14/2012. Compute the next year in which, if the date is written 03/14/$YYYY$, Pi Day will be celebrated in a year that is divisible by 7 and the sum of all eight digits is divisible by 7.

R2-2. *Let N be the number you will receive.* Let L be the sum of the tens and units digits of N. Let W be the two-digit integer formed by the leftmost two digits of N. Each layer of my birthday cake is 1 inch thick. Each layer is 2 inches wider than the layer immediately above it. Each layer is centered on the layer below it. A single candle is centered on the top layer as shown in the diagram (for a 4-layer cake). If the cake has L layers and the bottom layer is W inches wide, the candle is T inches tall. Compute T.



R2-3. *Let N be the number you will receive.* An arithmetic progression has 4 terms, the first term is N. If the sum of these four terms is 314, compute the last (i.e. fourth) term in the arithmetic progression.

2012 Relay Answers

R1-1. 216

R1-2. 24

R1-3. 17

R2-1. 2121

R2-2. 8.5

R2-3. 148.5

2012 Relay Solutions

R1-1. This is the merging of three arithmetic series: $1+4+7+10+13+16 = \frac{6}{2}(1+16) = 51$, $2+6+10+14+18+22 = \frac{6}{2}(2+22) = 72$, and $3+8+13+18+23+28 = \frac{6}{2}(3+28) = 93$, so our desired sum is $51+72+93 = \mathbf{216}$.

R1-2. Let the problem situation be modelled by the system $3t+4f = N$ and $t = f+16$. By substitution, we have $3(f+16)+4f = N$, or $f = \frac{N-48}{7}$. Since $N = 216$, we have $f = 24$. There are 40 three-leaf clovers and **24** four-leaf clovers.

R1-3. M, A, and T must be non-collinear. Thus, we solve $\frac{8-4}{5-2} = \frac{N-4}{k-2}$, so $\frac{4}{3} = \frac{N-4}{k-2}$, which simplifies to $4k-8 = 3N-12 \rightarrow k = \frac{3N-4}{4}$. Substituting $N = 24$, we have $k = \mathbf{17}$.

R2-1. Let $ABCD$ denote the year. Then: $8+(A+B+C+D) = 7k$ and k must be at least 2. Thus, $(A+B+C+D) > 6$ and for the next few years, $(A,B) = (2,0)$, so we start with $C+D = 4$. We examine 2013, 2022, 2031 and 2040. All these satisfy the sum of the digits requirement, but fail the divisibility by 7 test. So we, try $(A,B) = (2,1)$, $C+D = 3$. Testing 2103, 2112, 2121, 2130, we quickly realize **2121** is the next year.

R2-2. The indent as we add layers to the cake is 1 unit over and 1 unit up, forming isosceles triangles on the left and right sides. In fact, the big triangle is isosceles $(45°-45°-90°)$ and the distance from vertex to base is T, the height of the candle, plus L, the number of layers. Therefore, $2(T+L) = W+2$ and $T = \frac{W+2-2L}{2} = 1-N+\frac{W}{2}$. Waiting, $N = 3$, $W = 21 \rightarrow T = \mathbf{8.5}$.

R2-3. Let a by the first term of the progression and d be the common difference. Then the 4 terms are a, $a+d$, $a+2d$ and $a+3d$, so $4a+6d = 314$, $d = \frac{314-4a}{6} = 52+\frac{1-2a}{3}$ and the fourth term in the arithmetic progression is $a+3d = a+[3(52)+(1-2a)/3] = 157-a$. Waiting, $a = 8.5$, so our answer is $\mathbf{148.5}$.

2012 Tiebreaker Problems

TB-1. Consider a deck with 2012 numbered cards, each containing exactly one integer from 1 to 2012. The cards are dealt in the following way: the top card is dealt onto a table, then the next card is put on the bottom of the pile in the dealer's hand, then the next card is dealt on the table, then the next card is put on the bottom of the pile, and so on, alternating between dealing a card on the table and putting the next one on the bottom of the pile. Suppose that, before the dealing begins, the cards are arranged in the deck so that the cards that are dealt on the table will land in the order $1, 2, 3, \ldots, 2012$. To do this, the first three cards in the deck must have 1, 1007, 2. Compute the number on the card that would have to be in the 2000th position.

TB-2. Right triangle BOX, with vertices $B(20, 12)$, $O(0, 0)$, and $X(20, 0)$, is reflected in its hypotenuse. If point Y is the image of point X, compute the slope of line segment $\overline{OY}$.

2012 Tiebreaker Answers

TB-1. 1886

TB-2. $\dfrac{15}{8}$

2012 Tiebreaker Solutions

TB-1. We imagine a deck numbered 1 through 2012, deal it in the described way, and look for where the card 2000 finishes up. The position in which the card 2000 finishes will tell us which card should be in the 2000th position.

We see that the first 1006 cards would be $1, 3, 5, 7, ..., 2011$. Then the next 503 cards are $2, 6, 10, ..., 2010$. The next 252 cards are $4, 12, 20, 28, ..., 2012$. Note here that the next card in the deck is 8, and that goes below the deck, so the next 125 cards are $16, 32, 48, ..., 2000$. Thus, we want to put card number $1006 + 503 + 252 + 125 = \mathbf{1886}$ in the 2000th position.

TB-2. If acute angle BOX is called θ, then the desired slope is equal to $\tan 2\theta$. We use the formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(\frac{3}{5})}{1-\left(\frac{3}{5}\right)^2} = \mathbf{\frac{15}{8}}$.

2013 Contest at Byram Hills High School (Westchester)

Contents

2013 Team Problems

T-1. Call the number 2013 a *0123-number* because it is formed using only the digits 0, 1, 2, and 3, and each of these digits is used at least once. Let G be the greatest 0123-number that is less than 2013, and let L be the least 0123-number that is greater than 2013. Compute $G + L$.

T-2. A pyramid has its vertex at the center of the face of a cube and its base coincides with the opposite face of the cube. If the volume of the region *inside* the cube but *outside* the pyramid is 1152 cubic units, compute the edge of the cube.

T-3. Consider the set of "Pythagorean" ordered triples (a, b, c) where $a \leq b \leq c$ and $a^2 + b^2 = c^2$ for integers a, b, and c. Compute the number of Pythagorean ordered triples that have the number 130 as either a or b or c.

T-4. The base-ten decimal expression of 2013! has many zeroes, N of which come consecutively as a string at the end of the numeral. Compute N.

T-5. The volume of a right circular cylinder is $k\pi$ cubic units, where $k > 1$. The sum of the areas of its bases exceeds its lateral surface area by $k\pi$ square units. Compute all possible integer values of k.

T-6. Let $f(p)$ be the number of digits in a minimum cycle in the decimal expansion of the prime $1/p$ for any prime $p \neq 2, 5$. That is, if $n = f(p)$, then n is the smallest integer such that $\frac{1}{p} = 0.\overline{a_1 a_2 \ldots a_n}$. Compute the smallest n for which there is more than one prime p for which $f(p) = n$.

T-7. Given $\triangle ABC$ with D on $\overline{AB}$ and E on $\overline{AC}$ such that $\overline{CD}$ bisects $\angle ACB$ and $\overline{BE}$ bisects $\angle ABC$. If $AD = 3$, $AE = 4$, and $EC = 8$, compute BC.

T-8. Suppose that the 100 United States Senators have decided to play musical chairs in the following way. 100 chairs are lined in a row in the hall outside the Senate chamber. The senators line up in alphabetical order and march around the chairs until the music stops. The first time the music stops, the senator at the beginning of the line sits down in any chair he or she chooses. As soon as the first senator is seated, the music resumes and so does the march of senators.

Every time the music stops, two rules are followed: (1) The senator at the head of the marching line sits down in a seat of his or her choosing and (2) If there is a senator in either of the seats adjacent to the senator who just sat down, this "neighbor" stands and moves to the end of the line of marching senators. *Only one "neighbor" must stand up if there are two "neighbors" for a just-seated senator.*

If this process continues indefinitely, compute the maximum number of senators that could be sitting at a moment when the music starts.

T-9. The polar coordinates of points N, Y, and S are $N(4, 140°)$, $Y(6\sqrt{2}, 50°)$, and $S(3, 20°)$. Compute the area of triangle NYS. *Note: If a point P has polar coordinates (r, θ), then P is r units from the origin O, and ray OP makes an angle of θ with the positive x-axis.*

T-10. Suppose that N points are evenly spaced on the circumference of a circle. Every point is connected to every other point by a line segment. Nonconsecutive points are connected in 7 times as many ways as consecutive points. Now, suppose it costs money to connect points. Consecutive point connections cost \$$B$ while nonconsecutive point connections cost \$$A$, where A and B are integers and $B > A > 0$. Compute the ordered pair (A, B) if B is as small as possible and the total connection cost is \$2550.

2013 Team Answers

T-1. 3351

T-2. 12

T-3. 8

T-4. 501

T-5. 4, 54, 343 in any order

T-6. 5

T-7. 12

T-8. 99

T-9. $\frac{33}{2}\sqrt{2} - 3\sqrt{3}$

T-10. $(18, 24)$

2013 Team Solutions

T-1. We have $G = 1320$ and $L = 2031$, so our answer is $1320 + 2031 = \mathbf{3351}$.

T-2. Let x denote the edge of the cube. Then the volume of the required region is $x^3 - \frac{1}{3} \cdot x^2 \cdot x = \frac{2}{3}x^3 = 1152 = 8(144)$, so $x^3 = 12^3$, and thus $x = \mathbf{12}$.

T-3. Any Pythagorean triple can be generated from integer p and q as follows: $p^2 - q^2$ or $2pq$ or $p^2 + q^2$. The question is: How can (p, q), where $p > q$, form a factor of 130, keeping in mind the fact that we can sometimes dilate a "primitive" triangle so that one of its sides is 130?

factor	$2pq$	$p^2 - q^2$	$p^2 + q^2$
130	$(65,1) = \{130,4224,4226\}$ $(13,5) = \{130,144,194\}$	no solution	$(11,3) = \{66,112,130\}$ $(9,7) = \{32,126,130\}$
65	no solution	$2\times(33,32) = \{130,4224,4226\}$ $2\times(9,4) = \{130,144,194\}$	$2\times(8,1) = \{32,126,130\}$ $2\times(7,4) = \{66,112,130\}$
26	$5\times(13,1) = \{130,840,850\}$	no solution	$5\times(5,1) = \{50,120,130\}$
13	no solution	$10\times(7,6) = \{130,840,850\}$	$10\times(3,2) = \{50,120,130\}$
10	$13\times(5,1) = \{130,312,338\}$	no solution	$13\times(3,1) = \{78,104,130\}$
5	no solution	$26\times(3,2) = \{130,312,338\}$	$26\times(2,1) = \{78,104,130\}$
2	no solution	no solution	no solution

Note that each triplet is represented twice in the table, so there are only **8** distinct Pythagorean triples with this property.

T-4. There are many more two's than five's in the factorization of 2013!, so we will count up the five's and know that each one will pair up with a two to give a terminal zero. There will be at least one factor of 5 in every number in the set $\{5, 10, 15, \ldots, 2010 = 5 \cdot 402\}$, so there are at least 402 terminal zeroes in 2013!. Further, there will be at least two factors of 5 in every number in the set $\{25, 50, 75, \ldots, 2000 = 25 \cdot 80\}$, at least three factors of 5 in every number in the set $\{125, 250, 375, \cdots, 2000 = 125 \cdot 16\}$, and at least four factors of 5 in every number in the set $\{625, 1250, 1875\}$. Our answer is $402 + 80 + 16 + 3 = \mathbf{501}$.

T-5. We have $2\pi r^2 = 2\pi rh + k\pi$ which implies $2r^2 = 2rh + k$. We also have $\pi r^2 h = k\pi$ which implies $k = r^2 h$. Substituting and simplifying, we have $2r = 2h + rh = h(r + 2)$. This means $h = \frac{2r}{r+2}$ and $k = \frac{2r^3}{r+2}$. We notice that if k is integral, then r is a power of 2. Trying $r = 2$, $r = 6$, and $r = 14$, we obtain $k = \frac{16}{4} = 4$, $k = \frac{412}{8} = 54$, and $k = \frac{2(14)^3}{2^4} = 7^3 = 343$. Are there any more? Dividing, $\dfrac{2r^3}{r+2} = 2r^2 - 4r + 8 - \dfrac{16}{r+2}$. If $r > 14$, the last term in the expression is non-integral; thus, there are only three values of k: **4**, **54**, and **343**.

T-6. Note that $0.\overline{a_1a_2\ldots a_n} = \dfrac{a_1a_2\ldots a_n}{\underbrace{99\cdots 9}_{n \text{ nines}}}$ must reduce to $\frac{1}{p}$, so p must be a prime factor of $\underbrace{99\cdots 9}_{n \text{ nines}}$. Now,

$$9 = 3^2,\ 99 = 3^2 \cdot 11,\ 999 = 3^3 \cdot 37,\ 9999 = 3^2 \cdot 11 \cdot 101,\ 99999 = 3^2 \cdot 41 \cdot 271$$

and

$$\frac{1}{3} = 0.\overline{3},\ \frac{1}{11} = 0.\overline{09},\ \frac{1}{37} = 0.\overline{027},\ \frac{1}{101} = 0.\overline{0099},\ \frac{1}{41} = 0.\overline{02439},\ \frac{1}{271} = 0.\overline{00369}.$$

So $f(p) = 1$ only when $p = 3$, $f(p) = 2$ only when $p = 11$, $f(p) = 3$ only when $p = 37$, $f(p) = 4$ only when $p = 101$, but $f(p) = \mathbf{5}$ when $p = 41$ or $p = 271$.

This problem was originally written by Dr. Leo J. Schneider, who wrote NYSML contests from 2001 until his death in 2010. We include this problem to honor his memory.

T-7. Since $\overline{BE}$ is an angle bisector, it splits the side of the triangle to which it is drawn in the ratio of the other two sides. Thus, $AB = 4x$ and $BC = 8x$ for some number x. Similarly, $AD = 12y$ and $DB = 8xy$ for some number y. We are given that $AD = 3$, so $y = 1/4$, and so $DB = 2x$. Thus, we can solve $AD + DB = AB$, or $3 + 2x = 4x$ to obtain $2x = 3$, so $BC = 8x = \mathbf{12}$.

T-8. Suppose the first k seats are filled with senators. If the next senator sits in chair $k+2$ or greater, nobody stands up. Continuing, if a senator sits in chair $k+1$ and the senator sitting in chair $k+2$ stands up, our fearless leaders will have filled the first $k+1$ seats. This is reminiscent of mathematical induction! This scenario could continue until 98 senators occupy the first 98 chairs. Then the 99th senator sits in the 100th chair, and the 100th senator must sit in a seat with at least one neighbor, so it is impossible for them all to be seated. The maximum is **99**.

T-9. The area of $\triangle NYS$ equals the area of $\triangle NOY$ plus the area of $\triangle YOS$ minus the area of $\triangle NOS$. Thus, the desired area is $\frac{1}{2} \cdot 4 \cdot 6\sqrt{2}\sin 90° + \frac{1}{2} \cdot 3 \cdot 6\sqrt{2}\sin 30° - \frac{1}{2} \cdot 4 \cdot 3 \sin 120°$, or $12\sqrt{2} + \frac{9}{2}\sqrt{2} - 3\sqrt{3} = \mathbf{\frac{33}{2}\sqrt{2} - 3\sqrt{3}}$.

This is a "NYSML Classic". It is very much like question T10 from NYSML1983. We think it's an oldie but a goodie!

T-10. There are N possible consecutive point connections (think about the sides of the N-gon) and there are $\frac{N(N-3)}{2}$ possible nonconsecutive point connections (the diagonals of the N-gon). Thus, $\dfrac{N}{\frac{N(N-3)}{2}} = \dfrac{2}{N-3} = \dfrac{1}{7}$ implies that $N = 17$ and $17B + 119A = 2550$ implies that $B = 150 - 7A$. Certainly, $(A, B) = (1, 143)$ satisfies the requirements but not with minimal

B. Since as A increases by 1, B decreases by 7, we require that after x decrements of 7 to B and x increments of 1 to A, $B = 143 - 7x > A = 1 + x \to 142 > 8x \to x \leq 17$. Thus, $A = 1 + 17 = 18$ and $B = 143 - 7(17) = 24$. The answer is $(\mathbf{18}, \mathbf{24})$.

2013 Individual Problems

I-1. Compute the units digit of 2013^{2013}.

I-2. Compute the number of ordered pairs (x, y) of integers that satisfy $0 < x < 10$ and $0 < y \leq 10$ and $|x - y| = 1$.

I-3. The quadratic equation $(2x - 1)(5x + 3) = 35$ has one integer solution and one non-integer solution. Compute the non-integer solution.

I-4. Compute

$$\sum_{k=1}^{2013} (-1)^k k^2.$$

I-5. Consider quadratic equations of the form $x^2 + Bx + C = 0$ where B is a one-digit odd positive integer and C is a two-digit odd positive integer. Of the $5 \cdot 45 = 225$ quadratic equations of this type, N have two integer solutions. Compute N.

I-6. Three of the vertices of a cube have coordinates $V(2, 3, 5)$, $E(5, 7, 5)$, and $R(5, 7, 0)$. Compute the surface area of the cube in square units.

I-7. Compute $\sqrt{100 \cdot 101 \cdot 102 \cdot 103 + 1}$.

I-8. A regular dodecagon (12-sided polygon) has a perimeter of 12. Compute its area.

I-9. Compute the following sum:

$$\sin(1^\circ)\cos(59^\circ) + \sin(2^\circ)\cos(58^\circ) + \sin(3^\circ)\cos(57^\circ) + \cdots + \sin(59^\circ)\cos(1^\circ).$$

I-10. The zeroes of $f(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$ form an arithmetic progression of positive integers whose average is 2013. For all possible values of the coefficients A, B, C, D, E, and F, compute the least possible zero of $g(x) = Fx^5 + Ex^4 + Dx^3 + Cx^2 + Bx + A$.

2013 Individual Answers

I-1. 3

I-2. 17

I-3. $\dfrac{19}{10}$ or 1.9

I-4. -2027091

I-5. 0

I-6. 150

I-7. 10301

I-8. $3\sqrt{3}+6$

I-9. $\dfrac{59}{4}\sqrt{3}$

I-10. $\dfrac{1}{4025}$

2013 Individual Solutions

I-1. We notice that as we raise 2013 to successively larger powers, the units digit follows a predictable pattern: 3, 9, 7, 1, 3, 9, 7, 1, ... Since $2013 = 503 \cdot 4 + 1$, 2013^{2013} has the same units digit as 2013^1. Our answer is **3**.

I-2. The equation $|x - y| = 1$ tells us that the x- and y-coordinates differ by ± 1. This will usually mean that an x-value can be paired with exactly 2 y-values, one higher and one lower. This is the case for $2 \le x \le 9$, but not for $x = 1$. Thus, there are $8 \cdot 2 + 1 = \mathbf{17}$ ordered pairs (x, y).

I-3. The equation $(2x - 1)(5x + 3) = 35$ is equivalent to $10x^2 + x - 38 = 0$. Since one solution is an integer, one factor is $(x \pm A)$ and the other is $(10x \mp B)$, where the signs are intentionally reversed. After some experimentation, we have $(10x - 19)(x + 2) = 0$, so the non-integer solution is $\mathbf{\frac{19}{10}}$ or **1.9**.

I-4. Since $(2j)^2 - (2j - 1)^2 = 4j - 1$, we have $\sum_{k=1}^{2013}(-1)^k k^2 = \sum_{j=1}^{1006}[(2j)^2 - (2j - 1)^2] - 2013^2$, which is $\sum_{j=1}^{1006}[4j - 1] - 4052169 = \left[4\frac{1006\cdot 1007}{2} - 1006\right] - 4052169 = \mathbf{-2027091}$.

Alternate Solution: Note that $\sum_{k=1}^{2013}(-1)^k k^2 = -1 + \sum_{j=1}^{1006}[(2j)^2 - (2j + 1)^2]$, which equals $-\{1 + \sum_{j=1}^{1006}[4j + 1]\} = -\{1 + \left[4\frac{1006\cdot 1007}{2} + 1006\right]\} = -2013 \cdot 1007 = -2027091$.

This problem was originally written by Dr. Leo J. Schneider, who wrote NYSML contests from 2001 until his death in 2010. We include this problem to honor his memory.

I-5. The roots are integers that multiply to $-C$, an odd integer. Thus, the roots must be each odd integers. However, if the roots are odd integers, their sum must be even, and that would make $-B$ even, which would make B even, contrary to assumption. Therefore, there are no quadratic equations of this type that have two integer solutions. Our answer is $N = \mathbf{0}$.

This is a "NYSML Classic". It is very much like question I7 from NYSML2003. Math never goes bad!

I-6. Let's compute the distances between each pair of the given vertices. Notice that $VE = \sqrt{(5-2)^2 + (7-3)^2 + (5-5)^2} = 5$, $VR = \sqrt{(5-2)^2 + (7-3)^2 + (0-5)^2} = \sqrt{50} = 5\sqrt{2}$, and $ER = 5$ (by counting boxes). Since the three lengths satisfy the Pythagorean Theorem, $\overline{VR}$ is the hypotenuse of a right triangle with right angle at E, and so the side length of the cube is $VE = 5$. The surface area of the cube is $6 \cdot 5^2 = \mathbf{150}$.

I-7. Let $x = 100$. Then the desired quantity is $\sqrt{x(x+1)(x+2)(x+3)+1}$, which equals $\sqrt{x(x+3)(x+1)(x+2)+1} = \sqrt{(x^2+3x)(x^2+3x+2)+1}$, which is $\sqrt{(x^2+3x)^2+2(x^2+3x)+1}$, and this can be factored as $\sqrt{(x^2+3x+1)^2} = x^2+3x+1$. Substituting $x = 100$, we have our value of **10301**.

I-8. The dodecagon can be split into 12 congruent $30° - 75° - 75°$ triangles. In each of these triangles, the lengths of the congruent sides is $s = 2\sin 75°$, so the area of each of the 12 triangles is $A = \frac{1}{2} \cdot (4\sin^2 75°) \cdot \frac{1}{2} = \sin^2 75°$. Because we know that $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, we have $\sin^2 75° = \dfrac{\sqrt{3}+2}{4}$, and the area of the dodecagon is $12 \cdot \dfrac{\sqrt{3}+2}{4} = \mathbf{3\sqrt{3}+6}$.

I-9. Consider the first and last terms of this sum; they add to $\sin(60°) = \dfrac{\sqrt{3}}{2}$ (consult your favorite text on sum and difference trig formulas). So do the second and second-to-last, and the third and the third-to-last, and so on. There are 29 such pairings, and one term with no partner – $\sin(30°)\cos(30°) = \dfrac{\sqrt{3}}{4}$. The desired sum is $\dfrac{29\sqrt{3}}{2} + \dfrac{\sqrt{3}}{4} = \mathbf{\dfrac{59}{4}\sqrt{3}}$.

I-10. When the set of coefficients of a polynomial are reversed in order, the new roots are the reciprocals of the original roots. Seeing this, what is the reciprocal of the largest possible root of $f(x)$? Whatever the common difference d is, the roots of $f(x)$ are the integers $2013-2d$, $2013-d$, 2013, $2013+d$, and $2013+2d$. The least positive integer that $2013-2d$ could be occurs if $d = 1006$ and $2013 - 2(1006) = 1$. Thus, $2013 + 2(1006) = 4025$. Thus, the least possible root of $g(x)$ is $\mathbf{\dfrac{1}{4025}}$.

Power Question 2013: Elections

If only two candidates run for an office, it is straightforward to determine the results of an election: the one with the majority of votes. However, if there are more than two candidates, the determination is not so straightforward. For example, in a race with three candidates, suppose that one candidate receives one more than one third of the votes, one received one third and the third candidate receives one less than one third of the votes. The candidate who received the most votes was disapproved by almost two thirds of the electorate. With more candidates, the situation could become even worse. This problem will explore voter profiles. A **voter profile** is an individual voter's ranking of the candidates. We allow for indifference; that is, a voter may rank one candidate higher than all others but may be indifferent to the ranking of others. This question will explore the number of voter profiles among n candidates. We will not address the issue how to use the profiles to determine a winner in an election. There are several competing methods, all of which have some drawbacks. The celebrated Arrow's Theorem states that there does not exist a system that can meet certain naturally desired properties, such as lack of a dictator and others. First some terminology.

$\mathbf{P_n}$ The number of possible voter profiles with n candidates.

tiers In a profile, a tier is a set of candidates that the voter ranks equally. Within a tier, the order the candidates are listed is immaterial. For example, with three candidates the profiles $A/B, C$ and $A/C, B$ are the same. The notation gives an example of a two-tier profile with three candidates. Candidate A is ranked highest, and the voter is indifferent to candidates B and C, but ranking them both lower than A.

$\mathbf{P_{n,k}}$ The number of possible profiles with n candidates using k tiers. We note that a tier must contain at least one candidate. We assume throughout the problem that n and k are positive integers and that $k \leq n$.

An example

$P_3 = 13$ as shown below.
One-tier profiles: A, B, C
Two-tier profiles: $A/B, C$, $B/A, C$, $C/A, B$, $A, B/C$, $A, C/B$, $A, B/C$
Three-tier profiles $A/B/C$, $A/C/B$, $B/A/C$, $B/C/A$, $C/A/B$, $C/B/A$

The table also shows that $P_{3,1} = 1$, $P_{3,2} = 6$, and $P_{3,3} = 6$.

Binomial and Multinomial Coefficients

Certain ideas relating to binomial and multinomial coefficients apply to this situation. If $(X+Y)^n$ is expanded, the coefficient of the $X^k Y^{n-k}$ term is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. If $X = Y = 1$, the result is $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. (In other words, the nth row of Pascal's triangle sums to 2^n). This can be generalized. If $(X_1 + X_2 + \cdots + X_k)^n$ is expanded, the coefficient of the $X_1^{a_1} X_2^{a_2} \cdots X_k^{a_k}$ term is $\binom{n}{a_1, a_2, \cdots, a_k} =$

$\frac{n!}{a_1!a_2!\cdots a_k!}$. Similarly, letting $X_1 = X_2 \cdots = X_k = 1$ shows that $\sum_{a_1+a_2\cdots+a_k=n} \binom{n}{a_1,a_2,\cdots,a_k} = k^n$. The multinomial coefficient $\binom{n}{a_1,a_2,\cdots,a_k}$ also represents the number of ways of placing n objects into k boxes in such a way that box j gets a_j of the objects. Here the objects are the candidates and the boxes are the tiers. Thus there are k^n ways of placing n candidates into k boxes. However, not all of there will represent legitimate k-tier voter profiles since one or more of the tiers may be empty. If necessary, candidates can be referred to as C_1, C_2, etc., and tiers as T_1, T_2, etc.

P-1. Recall that $P_{3,1} = 1$. Show that $P_{n,1} = 1$ for all n. [4 pts]

P-2. It is true that $P_{3,3} = 6 = 3!$. Show that $P_{n,n} = n!$ for all n. [4 pts]

P-3. It is true that $P_{3,2} = 6 = 2^3 - 2$. Show that $P_{n,2} = 2^n - 2$. [4 pts]

P-4. Show that $P_n = \sum_{k=1}^{n} P_{n,k}$. [5 pts]

P-5. Provide an argument or list all possibilities to show that $P_{4,3} = 36$. [4 pts]

P-6. Show that $P_4 = 75$. [4 pts]

P-7. Show that $P_{n,3} = 3(3^{n-1} - 2^n + 1)$ for all n. [5 pts]

P-8. Show that $P_{n,n-1} = n!(n-1)/2$. A reasonable way to think about this is to think about placing the candidates in the tiers in such a way that no tier is empty. [5 pts]

P-9. Compute, with work, the value of P_5. [4 pts]

P-10. Show that $P_{n,k}$ satisfies the recursive relation $P_{n+1,k} = kP_{n,k} + kP_{n,k-1}$. [5 pts]

P-11. Prove that $P_{n,k} = k^n - \sum_{j=1}^{k-1} \binom{k}{j} P_{n,k-j}$. [6 pts]

Solutions to 2013 Power Question

P-1. Because all the candidates are on the same tier, and there is no importance in the order, there is only one way to place the n candidates.

P-2. Here, the order in which the n candidates are placed in the n tiers matters. This is equivalent to the fact that there are $n!$ ways to arrange the n candidates in a line.

P-3. For n candidates, there are 2^n ways of placing them into two tiers. But the two possibilities of putting all candidates in the first tier or all in the second tier must be eliminated. This leaves $2^n - 2$ possibilities.

P-4. Given n candidates, there must be 1 or 2 or ... or n tiers. For each $k = 1, 2, \cdots, n$, there are $P_{n,k}$ profiles.

P-5. There are $4 \times 3 \times 2$ ways to place one candidate onto each of the three tiers. The fourth candidate can be placed in three ways. This gives $24 \times 3 = 72$ possibilities. However, this must be divided by 2 to account for the two ways of placing the candidates on the one tier which has two candidates.

P-6. By **P-4**, $P_4 = P_{4,1} + P_{4,2} + P_{4,3} + P_{4,4} = 1 + 2^4 - 2 + 36 + 4! = 1 + 14 + 36 + 24 = 75$.

P-7. We know that $P_{n,3}$ is equal to 3^n minus the sum of all possibilities in which one or more tiers is empty. There are three ways to choose one empty tier. Then, the candidates can be placed into the two remaining tiers in $P_{n,2} = 2^n - 2$ ways. There are three ways to choose two empty tiers and only one way to place the candidates into the remaining tier. Thus, $P_{n,3} = 3^n - 3(2^n - 2) - 3$. Factoring out the common factor of 3 gives $3(3^{n-1} - 2^n + 2 - 1)$, which is the desired result.

P-8. Similarly to the argument in **P-5**, there are $n \times (n-1) \times (n-2) \times \cdots \times 2 = n!$ ways to place one candidate in each of $n-1$ tiers. There are then $n-1$ ways to place the last candidate. But this must be divided by 2 to account for the two ways to place the candidates on the tier with two candidates.

P-9. Using the result from **P-4** again, $P_5 = P_{5,1} + P_{5,2} + P_{5,3} + P_{5,4} + P_{5,5}$. Using previous work, each of these can be easily computed. $P_{5,1} = 1$, $P_{5,2} = 2^5 - 2 = 30$, $P_{5,3} = 3(3^4 - 2^5 + 1) = 150$, $P_{5,4} = 5! \times 4/2 = 240$, and $P_{5,5} = 5! = 120$. Summing gives $P_5 = 541$.

P-10. A candidate may be alone on a tier or on a tier with others. If the candidate C_{n+1} is not alone, take one of the $P_{n,k}$ k-tier profiles with n candidates and place C_{n+1} on one of the k-tiers. This can be done in $kP_{n,k}$ ways. If C_{n+1} is alone, take one of the $P_{n,k-1}$ $(k-1)$-tier profiles and insert a space for a new tier. This insertion can be done in k ways. Summing the two possibilities gives the result.

P-11. If empty tiers were allowed, there would be k^n possible ways to place the n candidates into the k tiers. To eliminate profiles with empty tiers, suppose there are j such tiers. There are $\binom{k}{j}$ ways to choose the empty tiers, and for each of these ways, there are $P_{n,k-j}$ ways of placing the n candidates on the $k-j$ tiers. This must be done for $j = 1, 2, \cdots, k-1$ and the result follows.

2013 Relay Problems

R1-1. If $(3x-1)$ is a factor of $15x^2+kx-4$, compute k.

R1-2. *Let N be the number you will receive.* Parallelogram $HELP$ has vertices at $H(1,0)$, $E(N,3)$, and $P(2,4)$. The equation of the line containing diagonal $\overline{HL}$ can be expressed as $Ax+By=C$ where A, B, and C are integers with no common factor other than 1. Compute $|C|$.

R1-3. *Let N be the number you will receive.* The positive difference between the roots of $x^2+Bx+C=0$ is N. If $C+B=155$, and if the roots are positive, compute the larger root of the quadratic equation.

R2-1. All of the letters of the word ABSOLUTELY are to be arranged such that exactly one vowel and no other letter is sandwiched between the two L's. This can be done in $2^A \cdot B!$ ways (where A and B are positive integers and B is as small as possible). Compute the ordered pair (A,B). *Do not consider Y a vowel.*

R2-2. *Let (A, B) be the ordered pair you will receive.* Compute the sum of <u>all</u> values of x for which $|2x-A|=B-x$. If the equation $|2x-A|=B-x$ has no solutions, pass back 0.

R2-3. *Let N be the number you will receive.* Compute all values of x such that

$$|x-N|=10-|x-N-1|.$$

2013 Relay Answers

R1-1. 7

R1-2. 1

R1-3. 14

R2-1. $(5, 7)$

R2-2. 2

R2-3. -2.5, 7.5 or equivalent simplified answers, in either order

2013 Relay Solutions

R1-1. The other factor must be of the form $(5x + B)$, and since $-1 \cdot B = -4$, we have $B = 4$ and $(3x - 1)(5x + 4) = 15x^2 + 7x - 4$. Our answer is $\mathbf{7}$.

R1-2. The slope of $\overline{HE}$ is the same as the slope of $\overline{LP}$, so we substitute and equate $\dfrac{3-0}{7-1} = \dfrac{4-y}{2-x}$ where (x, y) are the coordinates of L. This tells us that $8 - 2y = 2 - x \to x - 2y = -6$. We also have that the slope of $\overline{HP}$ is the same as the slope of $\overline{LE}$, so we substitute and equate $\dfrac{4-0}{2-1} = \dfrac{y-3}{x-7}$ where (x, y) are the coordinates of L. This tells us that $4x - 28 = y - 3 \to 4x - y = 25$. This system solves to give us $L(8, 7)$. The equation of the line we want is $y - 0 = 1(x - 1)$, or $x - y = 1$. Our answer is $\mathbf{1}$.

R1-3. Let the roots be r and $r-N$. We have $C+B = r(r-N)-(r+r-N) = r^2-(2+N)r+N = 155$. Substituting, this is equivalent to $r^2 - 3r - 154 = 0$. Factoring gives us $(r - 14)(r + 11) = 0$, whose only positive root is $r = \mathbf{14}$.

R2-1. The three-letter combination LXL occupies three consecutive positions, and these positions could begin anywhere from the first slot to the eighth slot, which gives a total of 8 possibilities. The wildcard X could be any of the four vowels. The remaining 7 letters are all different, so they can fill the remaining spaces in 7! ways. Thus, there are $4 \cdot 8 \cdot 7! = 2^5 \cdot 7!$ ways to arrange the letters satisfying the given conditions. Pass back $(\mathbf{5}, \mathbf{7})$.

R2-2. First, we must assume that $B - x$ is nonnegative, or else the equation cannot have a solution at all. So, with this assumption, we have $2x - A = \pm(B - x)$, and thus $x = \frac{A+B}{3}$ or $x = A - B$. The sum of these is $\dfrac{4A - 2B}{3}$. The answer must be 0 or $\dfrac{4A-2B}{3}$. For $(A, B) = (5, 7)$, we have $x = 4$ or $x = -2$, and both of those produce a non-negative value for $B - x$. Thus, our sum is $4 - 2 = \mathbf{2} = \dfrac{4 \cdot 5 - 2 \cdot 7}{3}$.

R2-3. Our equation is equivalent to $|x - N| + |x - (N + 1)| = 10$. Since N and $N + 1$ are consecutive integers, they are coordinates of two points on the number line 1 unit apart; call them P and Q. Let x be the coordinate of a generic point R on the number line. Then, our equation is equivalent to $RP + RQ = 10$. If R were between P and Q, then $RP + RQ = 1$; thus, no point between P and Q could be a solution. If R is to the left of P, we have $N - x + (N + 1) - x = 10 \to x = \frac{2N-9}{2}$. If R is to the right of Q, we have $x - N + x - (N + 1) = 10 \to x = \frac{2N+11}{2}$. Substituting, we have $x = \mathbf{-5/2}$ or $x = \mathbf{15/2}$.

2013 Tiebreaker Problems

TB-1. The sum of the infinite series $\frac{1}{2013} + \frac{2}{2013^2} + \frac{3}{2013^3} + \cdots + \frac{n}{2013^n} + \cdots$ can be written as $\frac{A}{B^2}$, where the fraction is in lowest terms. Compute the sum of the infinite series in this form.

TB-2. Three of the vertices of a cube have coordinates $C(4,7,9)$, $U(5,-2,5)$, and $B(8,6,0)$. Compute the volume of the cube in cubic units.

2013 Tiebreaker Answers

TB-1. $\dfrac{2013}{2012^2}$

TB-2. 343

2013 Tiebreaker Solutions

TB-1. We may look at this series as an infinite sum of infinite series: $(\frac{1}{2013}+\frac{1}{2013^2}+\frac{1}{2013^3}+\cdots+\frac{1}{2013^n}+\cdots)+(\frac{1}{2013^2}+\frac{1}{2013^3}+\frac{1}{2013^4}+\cdots+\frac{1}{2013^{n+1}}+\cdots)+(\frac{1}{2013^3}+\frac{1}{2013^4}+\frac{1}{2013^5}+\cdots+\frac{1}{2013^{n+2}}+\cdots)+\cdots$. This becomes $\frac{\frac{1}{2013}}{1-\frac{1}{2013}}+\frac{\frac{1}{2013^2}}{1-\frac{1}{2013}}+\frac{\frac{1}{2013^3}}{1-\frac{1}{2013}}+\cdots$, which is equivalent to $\frac{1}{2012}+\frac{1}{2013\cdot 2012}+\frac{1}{2013^2\cdot 2012}+\cdots$, or $\frac{1}{2012}\left(1+\frac{1}{2013}+\frac{1}{2013^2}+\cdots\right)$, which has sum $\mathbf{\frac{2013}{2012^2}}$.

TB-2. Let's compute the distances between each pair of the given vertices. Note that

$$CU=\sqrt{(5-4)^2+(-2-7)^2+(5-9)^2}=\sqrt{98},$$

$UB=\sqrt{(8-5)^2+(6-(-2))^2+(0-5)^2}=\sqrt{98}$, and $CB=\sqrt{(8-4)^2+(6-7)^2+(0-9)^2}=\sqrt{98}$. Because these three vertices form an equilateral triangle, it must be that each pair of these given vertices are opposite vertices of a face of the cube. Because the diagonal of one face of the cube measures $\sqrt{98}=7\sqrt{2}$, we have that the edge length of the cube is $\frac{7\sqrt{2}}{\sqrt{2}}=7$. The volume of the cube is $7^3=\mathbf{343}$.

2014 Contest at Fayetteville-Manlius High School (Onondaga)

Contents

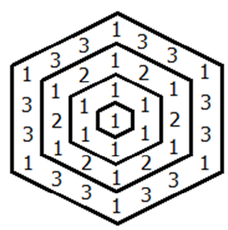
2014 Team Problems

T-1. In a box with 10 blocks, 6 are identical red blocks and 4 are identical blue blocks. Chris is going to make a tower of these 10 blocks, stacking one on top of the other until the tower is 10 blocks high. Compute the number of distinct towers Chris can make.

T-2. For positive integer values of x, there are exactly 5 integers between x and $x+6$, namely $x+1$, $x+2$, $x+3$, $x+4$, and $x+5$. However, the number of integers between x^2 and $(x+6)^2$ depends on the value of x. In terms of x, find a simplified expression for the number of integers strictly between x^2 and $(x+6)^2$.

T-3. The value of $2^{2014} + 2^{2017} + 2^n$ is a perfect square for one integer $n \geq 2017$. Compute this value of n.

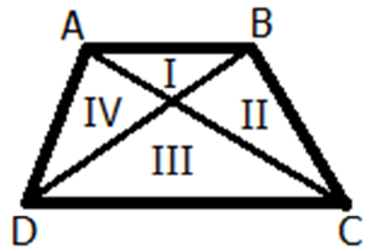
T-4. Pascal's Hexagon consists of concentric hexagons each containing 6 overlapping copies of Pascal's Triangle, except for the innermost hexagon which contains a single 1. A Level 3 Pascal's Hexagon is shown. If the pattern were to be continued to create a Level 7 Pascal's Hexagon, the sum of all of the numbers in the Hexagon would be P. Compute P.



T-5. Compute the number of integers in $\{10000, 10001, 10002, \cdots, 99999\}$ that are palindromes and multiples of 99.

T-6. Any circle in the (x, y)-plane that passes through $(6, -8)$ and $(8, -2)$ cannot also pass through the point $(K, 2014)$. Compute K.

T-7. In trapezoid $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ are drawn, splitting it into four regions (numbered I, II, III, and IV). The area of each numbered region is a positive integer. The area of region II is 2014. Compute the greatest possible difference between any two of these four numbered areas.



T-8. Larry and Gil each select a number from the set $\{1, 2, \cdots, 12, 13\}$, raise their numbers to the 2014th power, and then add the results. They find that the last four digits of their sum are 9952. Given that the numbers that Larry and Gil selected were L and G, and given that $L < G$, compute the ordered pair (L, G).

T-9. Let $f(x) = 5x + 1$, $f^2(x) = f(f(x))$, and so on, so that in general, $f^{n+1}(x) = f(f^n(x))$ for natural numbers $n \geq 1$. Compute the greatest prime factor of $f^7(31)$.

T-10. Given a regular n-gon, if k more sides were added to produce a regular polygon with $(n + k)$ sides, where $k \geq 1$, then the measure of each interior angle would increase by $(k + 3)$ degrees. Compute the sum of the four possible values of k such that $k < 30$.

2014 Team Answers

T-1. 210

T-2. $12x + 35$

T-3. 2018

T-4. 1483

T-5. 2142

T-6. 680

T-7. 4056195

T-8. $(4, 6)$

T-9. 521

T-10. 43

2014 Team Solutions

T-1. Choose the locations for the 4 blue blocks; this will determine the locations for the red blocks. Therefore, we seek $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \mathbf{210}$.

T-2. The smallest integer in the specified interval is $x^2 + 1$. The largest integer is $(x + 6)^2 - 1 = x^2 + 12x + 35$. Subtracting x^2 from each, we count the integers from 1 to $12x + 35$ inclusive. There are $\boldsymbol{12x + 35}$ such integers.

T-3. If $2^{2014} + 2^{2017} + 2^n = 2^{2014}(1 + 2^3 + 2^{n-2014}) = 2^{2014}(9 + 2^{n-2014})$ is a perfect square, then $9+2^{n-2014}$ is a perfect square. Notice that $9+16 = 25$ is a perfect square. For a more thorough solution, consider $9 + 2^x = y^2$; this implies $2^x = (y - 3)(y + 3)$, so $y - 3$ and $y + 3$ are powers of 2 that are 6 units apart, or 2 and 8. Thus, $y = 5$ and $x = 3$. Therefore, $n - 2014 = 4$ and $n = \mathbf{2018}$.

T-4. The nth row of Pascal's Triangle adds up to 2^n. Thus, the sum of all of Pascal's Triangle up to and including Row n will be $2^{n+1} - 1$. However, to make Pascal's Hexagon, the 1s along the edges get counted in two Triangles each (so subtract them out), and the 1 in the center is both included and excluded for all six Triangles (so add it back in). That is a grand total of $6((2^{n+1} - 1) - (n + 1)) + 1$. For $n = 7$, that sum is $6(255 - 8) + 1 = \mathbf{1483}$.

T-5. Since $abcba = 10001a + 1010b + c = (9999a + 990b) + (2a + 20b + c)$ and a, b, and c are digits with $a > 0$, it follows that $abcba$ will be divisible by 99 if and only if $2a + 20b + c = 99$ or 198. The fact that $2 \leq 2a + c \leq 27$ implies that $b = 4$ and $2a + 20b + c = 99$ or $b = 9$ and $2a + 20b + c = 198$. When $b = 4$ we have $2a + c = 19$ which yields $abcba = 94149$, 84348, 74547, 64746, and 54945. When $b = 9$ we have $2a+c = 18$ which yields $abcba = 99099$, 89298, 79497, 69696, and 59895. The answer is **10**.

This question was authored by Dr. Leo J. Schneider, who passed away in 2010. Dr. Schneider was the main author of NYSML from 2001-2010. We include this question to honor his memory.

T-6. Any three points determine a circle unless they are collinear. The line through $(6, -8)$ and $(8, -2)$ is $y = 3x - 26$. Solving $3K - 26 = 2014$ yields $K = \mathbf{680}$.

This is a "NYSML Classic". It is very much like question T7 from NYSML2004. We think it's an oldie but a goodie!

T-7. The ratio of the areas I:II:III:IV is $x^2 : xy : y^2 : xy$ for some values x and y. *(Consider why this is true... We know that the ratio I:III will be the square of the ratio of the sides; call those x and y. Then, consider I:II. They share an altitude, and their bases to that altitude are segments of diagonal AC that are in the proportion x:y, so the ratio of their areas is x:y. Similar proportions hold to obtain the desired result.)* If all four numbered areas need to have integer values, then $2014 = xy$ and the biggest difference will occur between III and I. The value of $y^2 - x^2$ will be greatest when $y = 2014$ and $x = 1$, so our answer is $2014^2 - 1^2 = (2015)(2013) = \mathbf{4056195}$.

T-8. Notice that $L^{2014} + G^{2014}$ is the sum of two perfect squares and that its units digit is 2. Since every perfect square ends in 0, 1, 4, 5, 6, or 9, L^{2014} and G^{2014} either both end in 1 or both end in 6. A perfect square ends in 6 if and only if its tens digit is odd (a result that the reader should prove to himself or herself). Thus, if L^{2014} and G^{2014} both end in 1, their tens digits are even, and therefore the tens digit of the sum will be even. Since the tens digit is odd, L^{2014} and G^{2014} both end in 6. The only digits satisfying the criteria are 4 and 6, so $(L, G) = (\mathbf{4}, \mathbf{6})$.

T-9. Notice that $f^2(1) = 31$, so $f^7(31) = f^9(1)$. Now, notice that $f^9(1) = 5^9 + 5^8 + 5^7 + \cdots + 5 + 1$, or $\dfrac{5^{10} - 1}{5 - 1}$, which is $\frac{1}{4} \cdot (5^5 + 1)(5^5 - 1) = \frac{1}{4} \cdot 3126 \cdot 3124 = 1563 \cdot 1562$. This factors as $3 \cdot 521 \cdot 2 \cdot 11 \cdot 71$. The greatest prime factor is $\mathbf{521}$.

T-10. The measure of an interior angle of a regular n-gon is $\dfrac{180(n-2)}{n}^{\circ}$. The measure of an interior angle of a regular $(n+k)$-gon is $\dfrac{180(n+k-2)}{n+k}^{\circ} = 180\left(1 - \dfrac{2}{n+k}\right)^{\circ}$. We require that $180\left(1 - \dfrac{2}{n+k}\right) - \dfrac{180(n-2)}{n} = k + 3$. This implies that $180 - \dfrac{360}{n+k} - 180 + \dfrac{360}{n} = k + 3 \Rightarrow 360\left(\dfrac{1}{n} - \dfrac{1}{n+k}\right) = k + 3$, which implies $\dfrac{360k}{n(n+k)} = k + 3$, so $n(n+k) = \dfrac{360k}{k+3}$. Now we know $n^2 + kn - C = 0$ where $C = \dfrac{360k}{k+3}$.
By the quadratic formula, we have $n = \dfrac{-k \pm \sqrt{k^2 + 4C}}{2}$ and since $n \geq 3$, $n = \dfrac{-k + \sqrt{k^2 + 4C}}{2}$.
Since the least possible interior angle of a regular polygon is 60° and the greatest possible interior angle is less than 180°, the difference is less than 120° and $4 \leq k + 3 < 120 \Rightarrow 1 \leq k < 117$. Thus, we require that $k^2 + 4C$ be a perfect square and we examine $(k, \sqrt{k^2 + 4C})$ for integer values of k between 1 and 30. The fruits of our labor are:
If $k = 1$, then $\sqrt{k^2 + 4C} = 19$ and $n = 9$.
If $k = 3$, then $\sqrt{k^2 + 4C} = 27$ and $n = 12$.
If $k = 12$, then $\sqrt{k^2 + 4C} = 36$ and $n = 12$.
If $k = 27$, then $\sqrt{k^2 + 4C} = 45$ and $n = 9$.
No other values of k generate integers n. Our sum is $1 + 3 + 12 + 27 = \mathbf{43}$.

2014 Individual Problems

I-1. A rectangular box has a volume of 2014 cubic inches. Each side has an integer length, and each side measures more than 1 inch. Compute the least possible area of one face of the solid.

I-2. A palindrome is a number that reads the same forwards and backwards. Call a palindrome n a *superpalindrome* if removing any two digits of n leaves a palindrome. Compute the number of superpalindromes between 100 and 10000.

I-3. Rhombus $RHOM$ has a perimeter of 24 cm. Square $ROSE$ has an area of 64 sq cm. Compute the area of rhombus $RHOM$ in sq cm.

I-4. Dora and Diego are solving a NYSML problem independently. The probability that Dora and Diego both solve the problem is 0.22. The probability that Dora or Diego (but not both) solves the problem is 0.51. Dora has a greater probability of solving the problem than Diego does. Compute the probability that Dora solves the problem.

I-5. In $\triangle ABC$, $AB = 4$ and $AC = 6$. D is the midpoint of $\overline{AC}$ and $BD = 3$. Compute the area of $\triangle ABC$.

I-6. A and B are two consecutive vertices in a convex n-gon (a polygon with n sides). If all diagonals <u>from these two vertices</u> are drawn, there are 406 points of intersection in the interior of the polygon. Compute n.

I-7. If $\log_2(\log_4(\log_{16}(x))) = 2$, compute $8^{\log_{32}(\log_2(x))}$.

I-8. Regular hexagon $HEXAGN$ is inscribed in circle O, and R is a point on minor arc HN of circle O. If $RE = 10$ and $RG = 8$, then RN can be expressed in the form $a\sqrt{b} + c$, where a, b, and c are integers. Compute RN in this form.

I-9. For every positive integer n, let $f(n)$ equal the sum of the cubes of the digits of n. For example, $f(1234) = 100$ since $1 + 8 + 27 + 64 = 100$. Compute the greatest n for which $f(n) > n$.

I-10. If $x^2 + x - 1 = 0$, compute all possible values of $\dfrac{x^2}{x^4 - 1}$.

2014 Individual Answers

I-1. 38

I-2. 99

I-3. $16\sqrt{5}$

I-4. 0.55 or $\dfrac{11}{20}$

I-5. $4\sqrt{5}$

I-6. 31

I-7. 64

I-8. $4\sqrt{3}-5$

I-9. 1999

I-10. $\pm\frac{\sqrt{5}}{5}$ (need both answers)

2014 Individual Solutions

I-1. Notice that $2014 = 2 \cdot 19 \cdot 53$. Thus, the faces have areas $2 \cdot 19$, $2 \cdot 53$, and $19 \cdot 53$. The smallest of these is $2 \cdot 19 = \mathbf{38}$.

I-2. Notice that any three-digit number is a superpalindrome since removing any two digits leaves a one-digit number (which is a palindrome). Since there are $9 \cdot 10 = 90$ three-digit palindromes, there are 90 three-digit superpalindromes. Now, if $abba$ is a four-digit superpalindrome, then ab must be a palindrome, which means $a = b$. Thus, a four-digit superpalindrome must be of the form $aaaa$, and there are 9 of those. Therefore, we have $90 + 9 = \mathbf{99}$ superpalindromes between 100 and 10000.

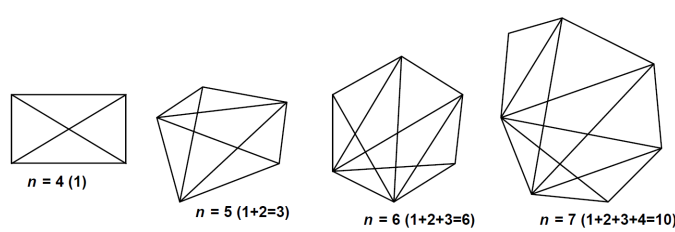
I-3. A side of $ROSE$ lies on the long diagonal of $RHOM$ and has length $\sqrt{64} = 8$. Thus, if both diagonals are drawn and intersect at X, $RX = \frac{1}{2}(8) = 4$ and $MX = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Thus, the area of $RHOM$ is four times the area of triangle RMX, or $4 \cdot \frac{1}{2} \cdot 2\sqrt{5} \cdot 4 = \mathbf{16\sqrt{5}}$.

I-4. Let the probability that Dora solves the problem be p and the probability that Diego solves the problem be q. Then $pq = 0.22$ and $p(1-q)+q(1-p) = 0.51 \rightarrow p+q-2pq = 0.51 \rightarrow p+q = 0.95$. There are two values that solve this system: $p = 0.4$ or $p = 0.55$. We choose $p = \mathbf{0.55}$.

This is a "NYSML Classic". It is very much like question T3 from NYSML1999. Math never goes bad!

I-5. We could proceed with Stewart's Theorem and Heron's Formula, but there may be an easier way. Notice that $\triangle ABD$ is isosceles. If we let E denote the midpoint of $\overline{AB}$, then $\overline{DE}$ is the altitude of $\triangle ABD$ and we can use the Pythagorean Theorem to find that $DE = \sqrt{5}$. Observe that $\triangle AED$ is similar to $\triangle ABC$ by a ratio of $1 : 2$. Thus, $BC = 2\sqrt{5}$ and the area of (right) triangle ABC is $\frac{1}{2} \cdot 4 \cdot 2\sqrt{5} = \mathbf{4\sqrt{5}}$.

I-6. Examine a few polygons with a small number of sides.



Draw all the diagonals from the first vertex A traversing the polygon in a clockwise direction to reach the opposite endpoints. Proceeding clockwise from A to the adjacent vertex B, draw the diagonals in a similar fashion. The first diagonal drawn intersects exactly one other diagonal. Each successive diagonal drawn intersects one more diagonal. Therefore, we must sum the first $(n-3)$ positive integers. This means $\frac{(n-3)(n-2)}{2} = 406$. Thus, we are looking for two consecutive integers whose product is 812. Take the square root of $812 \approx 900$ to get a starting approximation of 30. We see that $30 \cdot 29 = 870$, which is too big. But, $29 \cdot 28 = 812$ – Bingo! Thus, $n - 3 = 28 \rightarrow n = \mathbf{31}$.

I-7. Solving for x, we obtain $2^2 = \log_4(\log_{16}(x))$, which implies $4^4 = \log_{16} x$, which implies $16^{256} = x = 2^{1024}$. Therefore, $8^{\log_{32}(1024)} = 8^2 = \mathbf{64}$.

I-8. Since $\angle ERG$ is inscribed on a diameter of a circle, it is a right angle, and so $EG = \sqrt{10^2 + 8^2} = 2\sqrt{41}$. Thus, $OG = \sqrt{41}$, and since $\triangle ONG$ is equilateral, $NG = \sqrt{41}$. Consider $\triangle ENG$, which is right, so $EN = \sqrt{164 - 41} = \sqrt{123}$. Now, consider $\triangle ERN$. We know that $\angle ERN$ measures $120°$, so we apply the Law of Cosines to $\triangle ERN$: $123 = 10^2 + RN^2 - 2 \cdot 10 \cdot RN \cdot -\frac{1}{2}$, which gives us $RN^2 + 10RN - 23 = 0$, which can be solved to obtain $RN = \mathbf{4\sqrt{3} - 5}$.

I-9. The greatest n must have fewer than 4 digits since when $n \geq 4$, we have $n \cdot 9^3 < 10^n - 1$ making $f(n) < n$. In the following, let a, b, c, and d represent digits base 10, and $dcba$ will represent a four digit integer, occasionally with one or more of d, c, b, or a being constant.
When $9 \geq d > 2$, we have $f(dcba) \leq f(d999) = d^3 + 3(729) = d^3 + 2187 < 2999 < dcba$, so $n < 3000$.
If $d = 2$, then $f(2cba) \leq 8 + 512 + 2(729) = 1978 < 2cba$ if any one of $c, b, a < 9$. Furthermore, $f(2999) = 2195 < 2999$, so $n < 2000$.
Since $f(1999) = 2188 > 1999$, we have the largest $n = \mathbf{1999}$.

This problem was originally written by Dr. Leo J. Schneider, who wrote NYSML contests from 2001 until his death in 2010. We include this problem to honor his memory.

I-10. We could simply substitute the values of the roots into the expression and simplify, but consider the following. Notice that if $x^2 + x - 1 = 0$, then $x - \frac{1}{x} = -1$. Thus, $x^2 + \frac{1}{x^2} = 3$ and $x^4 + \frac{1}{x^4} = 7$. Finally, notice that $\left(x^2 - \frac{1}{x^2}\right)^2 = \left(x^4 + \frac{1}{x^4}\right) - 2 = 7 - 2 = 5$, so $x^2 - \frac{1}{x^2} = \pm\sqrt{5}$. Therefore, $\frac{x^2}{x^4 - 1} = \frac{1}{x^2 - \frac{1}{x^2}} = \frac{1}{\pm\sqrt{5}} = \pm\mathbf{\frac{\sqrt{5}}{5}}$.

Power Question 2014: Numbers of Various Sizes

In this question you will explore an extension of the real number system $\mathbf{R}$, called $\mathbf{R}^*$. The elements (numbers) of $\mathbf{R}^*$ are rational numbers; that is, $t = t(x) \in \mathbf{R}^*$ if $t = P(x)/Q(x)$ where P and Q are polynomials and Q is not identically 0. Since $r \in \mathbf{R}$ can be expressed as the rational function $r/1$, every real number is an element of $\mathbf{R}^*$.

We define an order on $\mathbf{R}^*$ by $a < b$ if there is a positive integer N such that $x > N \Rightarrow a(x) < b(x)$. Thus, for example, $1000x < x^2$ since $x > 1000 \Rightarrow 1000x < x^2$. A number s is called positive *small* if $0 < s < r$ for every real $r > 0$. Similarly, ℓ is positive *large* if $\ell > r$ for every real $r > 0$. Similar definitions hold for negative small and negative large numbers. Note that $\mathbf{R}$ has no small or large numbers. Non-zero numbers in $\mathbf{R}^*$ that are not small or large are called *medium*. An example of a small number is $1/x$ since if $N > 1/r$ and $x > N$, $1/x < r$. Since this holds for any r, $1/x$ is small. Similarly, x is large. Two numbers a and b are *equivalent*, written $a \equiv b$, if $a - b$ is small. It is easily shown that all small numbers are equivalent to each other and all small numbers are equivalent to 0.

In this question, s, s_1, s_2, and so on, will refer to small numbers, with similar conventions holding for ℓ (large numbers) and m (medium numbers). Many of the questions below can be handled by carefully using the definitions.

P-1. Show that $1/x^n$ is small for any positive integer n. [3 pts]

P-2. Show that x^n is large for any positive integer n. [3 pts]

P-3. $\mathbf{R}$ is called *Archimedian* because for any positive a and b, no matter how small a is and no matter how large b is, there is a positive integer n such that $an > b$. Show that $\mathbf{R}^*$ does not have this property if a is small and b is medium. [4 pts]

P-4. Prove that $\mathbf{R}^*$ has the following properties:

a. $s_1 + s_2$ is small. [3 pts]

b. $s_1 \cdot s_2$ is small. [3 pts]

c. $\ell_1 + \ell_2$ is large. [3 pts]

d. $\ell_1 \cdot \ell_2$ is large. [3 pts]

e. $m \cdot s$ is small. [3 pts]

f. $m + s$ is medium. [5 pts]

P-5. Show by example that the product of a large number and a small number can be large or small or medium. [3 pts]

P-6. A function $f(t)$ is called *continuous at a* if $f(a+s) \equiv f(a)$. Show that $f(t) = t^2$ is continuous at all medium numbers. [4 pts]

P-7. Show by example that $f(t) = t^2$ may be discontinuous at a large value of t. [3 pts]

P-8. We define the derivative of $f(t)$, written $f'(t)$, as $f'(t) = \dfrac{f(t+s) - f(t)}{s}$. Show that if $f(t) = t^2$, $f'(t) \equiv 2t$. [3 pts]

P-9. Show that $\sin(s) \equiv s$. You may use the fact that if t is in radians, $\lim_{t \to 0} \dfrac{\sin t}{t} = 1$, which means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |\frac{\sin t}{t} - 1| < \epsilon$. [3 pts]

P-10. Show that $(\sin(t))' = \cos(t)$. Recall that $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. You may assume that $\cos(s) \equiv 1$. [4 pts]

Solutions to 2014 Power Question

P-1. For $x > 1$, $1/x^n < 1/x$. Thus, since $1/x$ is small, so is $1/x^n$.

P-2. For $x > 1$, $x^n > x$, so x^n is large.

P-3. Let a be small and let r be given. Since a is small, for every n there is an x such that $a < r/n$. But then $an < r/n \times n = r$, and this means that an is small and thus cannot equal b.

P-4. a. For any r there is an x such that $s_1 < r/2$ and $s_2 < r/2$. Then $s_1 + s_2 < r/2 + r/2 = r$ and $s_1 + s_2$ is small.

b. By definition of small, there is an x such that both s_1 and s_2 are less than $\sqrt{r}$. Thus, $s_1 s_2 < \sqrt{r}\sqrt{r} = r$, showing that $s_1 \cdot s_2$ is small.

c. There is an x such that both l_1 and l_2 are greater than $r/2$, giving $l_1 + l_2 > r$ and so $l_1 + l_2$ is large.

d. There is an x such that both l_1 and l_2 are greater than $\sqrt{r}$, giving $l_1 \cdot l_2 > r$ and so $l_1 + l_2$ is large.

e. There is an x such that $s < r/m$, giving $ms < r$ and so ms is small.

f. A number is medium positive if there is an r such that there is an N with the property that for any $x > N$, $m < r$. Since s is small, there is an N such that for $x > N$, $s < 1/2$, so that $m + s < m + 1/2$. Choosing $r = m + 1/2$ shows that $m + s$ is medium.

P-5. If the large number is x and the small number is $1/x^2$, then their product is $1/x$ which is small. If the large number is x^2 and the small is $1/x$, their product is x which is large. If the large number is x and the small number is $1/x$, their product is 1 which is medium.

P-6. We have $(t + s)^2 = t^2 + 2ts + s^2$. If t is medium, then ts is small, and s^2 is always small, so $ts + s^2$ is small. But then t^2 and $(t + s)^2$ differ by a small number, showing that t^2 is continuous at medium numbers.

P-7. As in part **P-5**, if t is large, ts may not be small and thus the difference between t^2 and $(t+s)^2$ may not be small.

P-8. We have $\dfrac{f(t+s) - f(t)}{s} = \dfrac{t^2 + 2ts + s^2 - t^2}{s} = \dfrac{2ts + s^2}{s} = 2t + s \equiv 2t$.

P-9. By the definition of limit, for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |\frac{\sin t}{t} - 1| < \epsilon$. By definition of small, $|s| < \delta$ for every δ. Then $|\frac{\sin s}{s} - 1| < \epsilon$ for every ϵ. In other words, $|\frac{\sin s}{s} - 1|$ is small. Thus, $\dfrac{\sin s}{s} \equiv 1$ or $\sin(s) \equiv s$.

P-10. We have $\dfrac{\sin(t+s) - \sin(t)}{s} = \dfrac{\sin(t)\cos(s) + \cos(t)\sin(s) - \sin(t)}{s} = \dfrac{\sin t(\cos s - 1) + \cos t \sin s}{s}$.
This is equivalent to $\dfrac{\sin t \cdot 0 + \cos t \sin s}{s} \equiv \cos t$.

2014 Relay Problems

R1-1. Egyptians wrote fractions of the form $\frac{a}{b}$ where $a \neq 1$ as the sum of unit fractions. The fraction $\frac{7}{26}$ can be written as $\frac{1}{K} + \frac{1}{M}$ for positive integers K and M with $M \geq K$ in exactly one way. Compute $K + M$.

R1-2. *Let N be the number you will receive.* Let k be the greatest positive integer for which $\frac{N}{2^k}$ is an integer. A triathlete swims 2 miles in 2 hours, runs 10 miles in 75 minutes, and bikes 38 miles in k hours. His average speed is A miles per hour. Compute $\lfloor A \rfloor$, the greatest integer less than or equal to A.

R1-3. *Let N be the integer you will receive.* Compute the number of integer solutions to the system $|x - 3| \leq 3N$ and $|x + 1| > N$.

R2-1. Given x and y are positive integers. In $\triangle ABC$, $m\angle A = (2x)^\circ$, $m\angle B = (3x + 5)^\circ$, and $m\angle C = y^\circ$. Compute the number of distinct values of x for which $\triangle ABC$ is scalene.

R2-2. *Let N be the positive number you will receive.* Let $K = \dfrac{N}{3} + 4$. Mr. Regular K-gon says to Mr. Isosceles Triangle (all of whose angles have integer degree measures), "Any of my interior angles are 100 degrees greater than 4 times the measure of your acute vertex angle V." Compute V.

R2-3. *Let N be the number you will receive.* In isosceles triangle ABC, $m\angle A = k \cdot m\angle B$, where k and $m\angle B$ are integers and $k > 1$. Let S denote the sum of the measures of a base angle and the corresponding vertex angle of $\triangle ABC$. Compute the number of distinct non-similar triangles ABC for which $S > 7 \cdot N$.

2014 Relay Answers

R1-1. 56

R1-2. 8

R1-3. 32

R2-1. 33

R2-2. 14

R2-3. 17

2014 Relay Solutions

R1-1. Notice that $\frac{7}{26}$ is just greater than $\frac{1}{4}$. So, suppose that $\frac{7}{26} = \frac{1}{4} + \frac{1}{M} \Rightarrow \frac{1}{M} = \frac{7}{26} - \frac{1}{4} = \frac{14-13}{52} = \frac{1}{52}$. Thus, $K + M = 4 + 52 = \mathbf{56}$.

R1-2. Our triathlete covers 50 miles in $k + 3.25$ hours. His average speed A is $\frac{50}{k+3.25} = \frac{200}{4k+13}$ mph. Now, substitute the value of k, which we obtain by recognizing that $\frac{56}{2^3}$ is an integer but $\frac{56}{2^4}$ is not, so $k = 3$. Therefore, $A = \frac{200}{25} = 8$. Pass back $\mathbf{8}$.

R1-3. While we're waiting for numbers to come back, let's examine cases. We note that the system only has solutions if $N > 2$. The solution is $[3 - 3N, -N - 1) \cup (-1 + N, 3 + 3N]$. If $N = 3$, there are 12 such integers. If $N = 4$, there are 16 such integers. If $N = 5$, there are 20 such integers. In fact, there are always $4N$ such integers. Substituting, we have $4(8) = \mathbf{32}$.

R2-1. We have $5x + 5 + y = 180$ which implies $y = 175 - 5x = 5(35 - x)$, so x can be any integer from 1 through 34 inclusive. However, $\triangle ABC$ cannot be equilateral or isosceles. Assume $\angle A \cong \angle B$; this is impossible, since $2x = 3x + 5 \to x = -5$. Assume $\angle A \cong \angle C$; since $y = 2x$, we have $7x + 5 = 180 \to x = 25$. Thus, x cannot be 25. Now, assume $\angle B \cong \angle C$; since $y = 3x + 5$, we have $8x + 10 = 180 \to x = 85/4$, which is not an integer. Therefore, there are $34 - 1 = \mathbf{33}$ possible scalene triangles.

R2-2. Let A be the measure of any angle in the regular K-gon. Then $A = 4V + 100 \to V = \dfrac{A - 100}{4} = \dfrac{\frac{(K-2)(180)}{K} - 100}{4} = \dfrac{80K - 360}{4K} = 20 - \dfrac{90}{K}$. Thus, K must be a factor of $90 = 2^1 \cdot 3^2 \cdot 5^1$. That means there are $(1+1)(2+1)(1+1) = 12$ factors, but $K = 1$ and $K = 2$ are physically impossible. $K = 3$ results in a negative value of V, leaving 9 ordered pairs (K, V). Since $N = 33$, $K = 15$, and so $V = \mathbf{14}$.

R2-3. Let $m\angle B = x$. If B is the vertex angle, then $(2k + 1)x = 180 \to x = \frac{180}{2k+1}$. Since $2k + 1$ is odd, we are looking for an odd factor of 180 and there are six possibilities: 1, 3, 5, 9, 15, and 45. Two of these (1 and 3) correspond to $k = 0$ and $k = 1$, so there are four physical triangles represented, with $S = 108$ or 100 or 96 or 92. If B is a base angle, then $(k + 2)x = 180 \to x = \frac{180}{k+2}$. 180 has 18 factors, but 1, 2, and 3 are off the table since k must be greater than 1. The remaining 15 values give rise to 15 isosceles triangles different from the four already found (that proof is up to the reader). There are 19 possible sums, but 2 of them are less than $7 \cdot 14 = 98$, so we have $19 - 2 = \mathbf{17}$ distinct sums.

2014 Tiebreaker Problems

TB-1. In a certain country called NYSMLand, there are only three kinds of currency, the 3-nysml bill, the 5-nysml bill, and the 15-nysml bill. Toni buys a car in NYSMLand, and the car costs 1500 nysmls. She could pay for her car in many different ways. One way is to pay with 500 3-nysml bills. Another is to pay with 50 15-nysml bills and 150 5-nysml bills. Compute the number of distinct ways in which she could pay for her car.

TB-2. The sum $1! + 2! + 3! + \cdots + n! + \cdots + 2014!$ is written as a base-ten numeral. The numeral ends in the two digits AB. Compute the two-digit numeral AB.

2014 Tiebreaker Answers

TB-1. 5151

TB-2. 13

2014 Tiebreaker Solutions

TB-1. This problem is analogous to finding whole number solutions to $3x+5y+15z=1500$. Since 5, 15, and 1500 are each divisible by 5, $3x$ must also be divisible by 5, so x must be divisible by 5. Similarly, $5y$ must be divisible by 3, so y must be divisible by 3. Therefore, there exist X and Y such that $15X+15Y+15z=1500$, which is equivalent to $X+Y+z=100$. There are $\binom{102}{2}$ ways to solve this over the whole numbers. *Note: when you go home, Google "Balls and Urns"!* The answer is $\frac{102 \cdot 101}{2} = \mathbf{5151}$.

TB-2. For $n \geq 10$, the last two digits of $n!$ are 00. We must add only 1! through 9!, keeping track of only the last two digits. We add $01+02+06+24+20+20+40+20+80=...13$. The last two digits are **13**.

This is a "NYSML Classic". It is very much like question T5 from NYSML1979. Good problems are timeless!

2015 Contest at Syosset High School (Nassau)

Contents

2015 Team Problems

T-1. For two positive numbers x and y, we define their arithmetic mean as $\dfrac{x+y}{2}$, their geometric mean as $\sqrt{xy}$, and their harmonic mean as $\dfrac{2}{\frac{1}{x}+\frac{1}{y}}$. Suppose that two positive numbers have a geometric mean of 24 and a harmonic mean of 22. Compute their arithmetic mean.

T-2. If $(x+2z):(2y+z):(2x+y)=1:3:5$ and $x+y+z=18$, compute the value of z.

T-3. There are 52 balls in a box. Each ball has a number. Four of the balls are numbered 0, four are numbered 1, and so on, such that the highest number on a ball is 12 (and this occurs for four balls). Three balls are chosen from the box without replacement. Compute the probability that at least one ball will have a two-digit number.

T-4. The perimeter of regular dodecagon $DISCOUNTABLE$ is 60. Compute its area in the form $a+b\sqrt{c}$, where a, b, and c are integers, and c cannot be divided by the square of any prime.

T-5. Define an *up-down integer* as a five-digit natural number $d_1d_2d_3d_4d_5$ where $0<d_1<d_2<d_3$ and $d_3>d_4>d_5$. The least up-down integer is 12310 and the greatest is 78987. Compute the number of base-10 up-down integers.

T-6. Compute the ordered triple of positive integers (a,b,c) such that

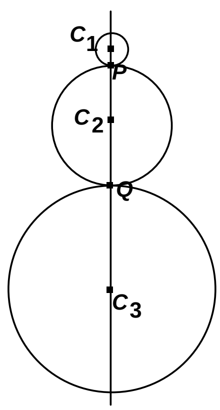
$$(\sqrt{2}+\sqrt{5}-\sqrt{7})(15\sqrt{a}+6\sqrt{b}+3\sqrt{c})=60.$$

T-7. Suppose that Priority Mail delivery costs \$5.60 and I want to use up stamps that I already have to pay the postage. I have large quantities of 3-cent, 34-cent, 39-cent, and 42-cent stamps in my desk. Compute the minimum number of stamps I could use to pay the \$5.60 charge.

T-8. In trapezoid $ABCD$, bases $\overline{AB}$ and $\overline{CD}$ have lengths 15 and 20 respectively. Points E and F are on $\overline{AD}$ and $\overline{BC}$ respectively such that $\overline{EF}\parallel\overline{AB}$. If $AE:ED=2:3$, compute EF.

T-9. The equation $\sum_{j=0}^{n}\left(\binom{n}{j}a_j\right) = (n+1)^2$ generates a system of equations for non-negative integers a_j and for integers n satisfying $0 \le n \le 2015$. Some of the 2016 equations in this system are $1a_0 = 1$, $1a_0 + 1a_1 = 4$, $1a_0 + 2a_1 + 1a_2 = 9$, $1a_0 + 3a_1 + 3a_2 + 1a_3 = 16$, and $1a_0 + 4a_1 + 6a_2 + 4a_3 + 1a_4 = 25$. Compute the sum $a_0 + a_1 + a_2 + ... + a_{2015}$.

T-10. Circles C_1, C_2, and C_3 have radii $r_1 = 1$, $r_2 = 3$, and $r_3 = 5$ respectively. Circles C_1 and C_3 are externally tangent to circle C_2 at points P and Q, respectively. Chord $\overline{MN}$ is drawn in C_3 perpendicular to line PQ at point R (where R is on line PQ). The lengths MN and C_3R are both integers. Compute C_1M such that $\triangle C_1MN$ has maximum perimeter.



2015 Team Answers

T-1. $\dfrac{288}{11}$

T-2. -2

T-3. $\dfrac{47}{85}$

T-4. $150 + 75\sqrt{3}$

T-5. 2142

T-6. $(2, 5, 70)$

T-7. 14

T-8. 17

T-9. 6

T-10. $\sqrt{241}$

2015 Team Solutions

T-1. Let the two numbers be x and y. Then $xy = 24^2$ and $\dfrac{2xy}{x+y} = 22$. Thus, $\dfrac{x+y}{2} = \dfrac{xy}{22} = \mathbf{\dfrac{288}{11}}$.

T-2. Let $x+2z = r$, $2y+z = 3r$, and $2x+y = 5r$. Adding these three equations yields $3(x+y+z) = 9r$. Substituting $x+y+z = 18$ yields $r = 6$. Subtract $2y+z = 18$ and $x+y+z = 18$ to obtain $x = y$. Solving now gives us $x = y = 10$ and $z = \mathbf{-2}$.

T-3. The desired probability is equal to 1 minus the probability of not drawing any ball with a two-digit number. This is $1 - \dfrac{\dbinom{40}{3}}{\dbinom{52}{3}} = 1 - \dfrac{40! \cdot 3! \cdot 49!}{3! \cdot 37! \cdot 52!} = 1 - \dfrac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50}$, which simplifies to $1 - \dfrac{38}{85} = \mathbf{\dfrac{47}{85}}$.

T-4. The regular 12-gon can be decomposed into 12 congruent isosceles triangles, and each of them can be bisected into right triangles by dropping the altitude from the vertex angle to the base. Let r represent the distance from the center of the 12-gon to the base angle of those isosceles triangles. Then, the right triangles have legs of length $r\sin(15^\circ)$ and $r\cos(15^\circ)$ (the latter is the altitude). From the perimeter, we know that $r\sin(15^\circ) = \dfrac{5}{2}$, or $r = \dfrac{10}{\sqrt{6}-\sqrt{2}}$. The area of each right triangle is $\frac{1}{2}(r\sin(15^\circ))(r\cos(15^\circ))$, making K, the area of the dodecagon, equal to $K = 12(r\sin(15^\circ))(r\cos(15^\circ)) = 6r^2\sin(30^\circ) = 6\left(\dfrac{10}{\sqrt{6}-\sqrt{2}}\right)^2 \cdot \dfrac{1}{2}$, which is $\dfrac{300}{8-4\sqrt{3}} = \mathbf{150 + 75\sqrt{3}}$.

T-5. Focus on d_3. It is not possible for d_3 to be 0 or 1 or 2. If $d_3 = 3$, there is $\dbinom{2}{2} = 1$ choice for the two digits on the left side of d_3 and $\dbinom{3}{2} = 3$ choices for the two digits on the right side of d_3. If $d_3 = 4$, there are $\dbinom{3}{2} = 3$ choices for the two digits on the left side of d_3 and $\dbinom{4}{2} = 6$ choices for the two digits on the right side of d_3. This pattern continues, so that the desired answer is $\dbinom{2}{2} \cdot \dbinom{3}{2} + \dbinom{3}{2} \cdot \dbinom{4}{2} + \cdots + \dbinom{8}{2} \cdot \dbinom{9}{2} = 1\cdot3 + 3\cdot6 + 6\cdot10 + 10\cdot15 + 15\cdot21 + 21\cdot28 + 28\cdot36 = \mathbf{2142}$.

T-6. Rewrite the given equation as $\dfrac{3}{\sqrt{2}+\sqrt{5}-\sqrt{7}} = \dfrac{15\sqrt{a}+6\sqrt{b}+3\sqrt{c}}{20}$. Then, rationalize the denominator by first multiplying the numerator and denominator on the left side by $(\sqrt{2}-\sqrt{5}-\sqrt{7})$ to obtain $\dfrac{3\sqrt{2}-3\sqrt{5}-3\sqrt{7}}{4-2\sqrt{14}}$, and then rationalize this fraction to obtain $\dfrac{15\sqrt{2}+6\sqrt{5}+3\sqrt{70}}{20}$. The desired ordered triple is $(\mathbf{2},\mathbf{5},\mathbf{70})$.

This is a "NYSML Classic". It is very much like question T7 from NYSML1980. We think it's an oldie but a goodie!

T-7. Notice that 40 is approximately the average value of a stamp (forgetting about the 3-cent stamps) so set 40 as the "target value" per stamp. This is especially useful since 40 divides 560. There are no 40-cent stamps, and each 42-cent stamp gives an "extra" 2 cents, so look to balance groups. For example, three 42-cent stamps plus one 34-cent stamp cost \$1.60 and one 42-cent stamp and two 39-cent stamps cost \$1.20. Can we use these groups to make a supergroup of 14 stamps? Yes! Use two of each group to obtain stamps worth \$5.60. Thus, the minimum number of stamps required is $8+4+2+0=\mathbf{14}$.
Note: 14 stamps is minimal; if 13 or fewer stamps were to be used, the total amount could be no more than $13 \cdot \$0.42 = \$5.46$.

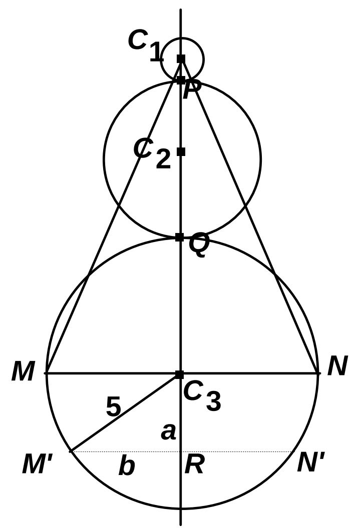
T-8. Extend $\overline{AD}$ and $\overline{BC}$ to meet at G. Then, triangles AGB, EGF, and DGC are similar. Thus, $\dfrac{AG}{AB}=\dfrac{EG}{EF}=\dfrac{DG}{DC}$. Note that $\dfrac{EG}{EF}=\dfrac{AG}{AB}=\dfrac{EG-AG}{EF-AB}=\dfrac{EA}{EF-15}$ and $\dfrac{DG}{DC}=\dfrac{EG}{EF}=\dfrac{DG-EG}{DC-EF}=\dfrac{DE}{20-EF}$. Therefore, $\dfrac{EA}{EF-15}=\dfrac{DE}{20-EF}$, or $\dfrac{EF-15}{20-EF}=\dfrac{EA}{DE}=\dfrac{2}{3}$, which implies $EF=\mathbf{17}$.

Alternate Solution: Extend $\overline{AD}$ and $\overline{BC}$ to meet at G as before, and note the similarity relations from the previous solution. Let $AE=2x$ so that $ED=3x$ and $AD=5x$. Because $\dfrac{AB}{CD}=\dfrac{3}{4}$, conclude that $\dfrac{AG}{DG}=\dfrac{AG}{AG+5x}=\dfrac{3}{4}$, hence $AG=15x$. Thus $\dfrac{AB}{EF}=\dfrac{AG}{EG}=\dfrac{15}{15+2}$, and because $AB=15$, it follows that $EF=\mathbf{17}$.

T-9. If $n \geq 3$, the value of a_n must be 0 because $\dbinom{n}{j}$ has a n^j term that will need to cancel. So, suppose that $a_0\dbinom{n}{0}+a_1\dbinom{n}{1}+a_2\dbinom{n}{2}=a_0(1)+a_1(n)+a_2(\frac{1}{2}n^2-\frac{1}{2}n)=n^2+2n+1$. For this to be an identity, $a_2=2$. Now, $a_0(1)+(a_1-1)n+n^2=n^2+2n+1$, so $a_1=3$, and $a_0=1$. This confirms that $a_n=0$ for $n\geq 3$ and the sum is $1+3+2+2012(0)=\mathbf{6}$.

T-10. Let chord $\overline{MN}$ be a diameter of C_3. Then, $C_1C_3 = 1+2(3)+5 = 12$ and $C_3M = 5 \rightarrow$Perimeter $= 2(13) + 10 = 36$. Can the perimeter be larger and MN and C_3R still be integers?

If $\overline{MN}$ moves closer to C_1, the perimeter of $\triangle C_1MN$ gets smaller. So consider the case where $\overline{MN}$ moves away from C_1. Because MN and C_3R are integers, the solutions (a, b) to $a^2 + b^2 = 5^2$ must be integral. Notice that the only positive integral solutions to $a^2 + b^2 = 5^2$ are $(3, 4)$ and $(4, 3)$.



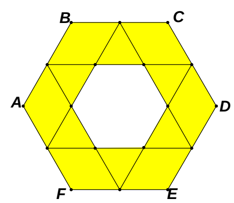
If $a = 3$, then $C_1M' = \sqrt{4^2 + 15^2} = \sqrt{241} \rightarrow$Perimeter $= 8 + 2\sqrt{241}$.
If $a = 4$, then $C_1M' = \sqrt{3^2 + 16^2} = \sqrt{265} \rightarrow$Perimeter $= 6 + 2\sqrt{265}$.
Is either of these values greater than 36? Since $15^2 = 225$ and $16^2 = 256$, $2\sqrt{241} + 8 > 2(15) + 8 = 38$. Is $2\sqrt{265} + 6 > 2\sqrt{241} + 8$? If this were true, then this would be equivalent to $\sqrt{265} - \sqrt{241} > 1$, which in turn is equivalent to $\dfrac{265 - 241}{\sqrt{265} + \sqrt{241}} = \dfrac{24}{\sqrt{265} + \sqrt{241}} > 1$. But the latter inequality is false because $\dfrac{24}{\sqrt{265} + \sqrt{241}} < \dfrac{24}{\sqrt{225} + \sqrt{225}} = \dfrac{24}{30} < 1$. Thus $2\sqrt{265} + 6$ must be less than $2\sqrt{241} + 8$. Therefore, the maximum perimeter of $\triangle C_1MN$ under these conditions occurs when $C_1M = \sqrt{\mathbf{241}}$.

2015 Individual Problems

I-1. Given that 2015 is the least of 2015 consecutive integers, compute the mean of these 2015 integers.

I-2. Compute the number of positive integers less than or equal to 2015 that share no positive factor with 2015 other than 1.

I-3. In the figure, $ABCDEF$ is a regular hexagon. Some midpoints of sides are connected to form a six-pointed star. Some of the hexagon is shaded. Compute the fraction of $ABCDEF$ that is shaded.



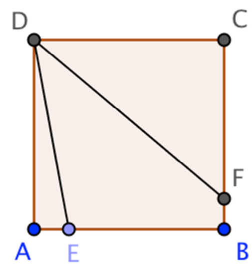
I-4. Rectangle $MATH$ is given, with $MA = 4$ and $AT = 6$. Equilateral triangles HIM and THE are constructed such that segments $\overline{MI}$, $\overline{HI}$, $\overline{TE}$, and $\overline{HE}$ are exterior to $MATH$. Compute the area of $\triangle AEI$.

I-5. Compute the shortest distance from the graph of $(x-4)^2 + (y-2)^2 = 4$ to the point $(7, 6)$.

I-6. The number $337^2 = 113569$ has 6 digits that are in non-decreasing order from left to right. The decimal number equivalent to $333...37^2$ has 2000 digits that also are in non-decreasing order from left to right. Compute the sum of those 2000 digits.

I-7. A very long natural number is created by writing the first 2015 natural numbers in a string. The number is 12345678910111213...20142015. Compute the 2015th (leftmost) digit of this number.

I-8. In square $ABCD$, E is on $\overline{AB}$ and F is on $\overline{BC}$ such that $\overline{DF}$ is an angle bisector of $\angle EDC$. Given that $DE = 20$ and $AD = 15$, compute $AE + CF$.



I-9. Consider a sequence $\{n_i\}$ for which $n_1 = 2$, $n_2 = 0$, $n_3 = 1$, $n_4 = 5$, $n_5 = 20$, $n_6 = 15$, and $n_i = n_{i-1} - n_{i-2} + n_{i-3} - n_{i-4} + n_{i-5} - n_{i-6}$ for $i \geq 7$. Compute n_{2015}.

I-10. Compute the number of positive integers n such that $n \leq 2015$ and n is divisible by $\lfloor \sqrt{n} \rfloor$, which is the greatest integer not exceeding $\sqrt{n}$.

2015 Individual Answers

I-1. 3022

I-2. 1440

I-3. $\dfrac{3}{4}$

I-4. $18 + 13\sqrt{3}$

I-5. 3

I-6. 6007

I-7. 0

I-8. 20

I-9. -15

I-10. 131

2015 Individual Solutions

I-1. Represent the 2015 consecutive integers as $x - 1007$, $x - 1006$, $\cdots$, $x + 1006$, and $x + 1007$. The sum of the integers is $2015x$, and their average is x. Since $x - 1007 = 2015$, $x = \mathbf{3022}$.

I-2. Note that $2015 = 5 \cdot 13 \cdot 31$. There are $2015 \div 5 = 403$ multiples of 5 which are not relatively prime to 2015. Similarly, there are $2015 \div 13 = 155$ multiples of 13 and $2015 \div 31 = 65$ multiples of 31. Thus, the number of positive integers less than 2015 that are relatively prime to 2015 is at least $2015 - 403 - 155 - 65 = 1392$. However, there are some numbers that have been counted twice in the subtraction – namely, the 31 multiples of 65, the 13 multiples of 155, and the 5 multiples of 403. Thus, the number of positive integers less than 2015 that are relatively prime to 2015 is at most $1392 + 31 + 13 + 5 = 1441$. However, 2015 has been discounted and recounted three times, and must be discounted. Our answer is **1440**.

I-3. Draw diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. The hexagon has now been divided into 24 congruent equilateral triangles, 18 of which are shaded. The required ratio is $\dfrac{18}{24} = \mathbf{\dfrac{3}{4}}$.

I-4. Note that each side of $\triangle AEI$ is the longest side of a triangle whose other sides are 4 and 6 and whose included angle measures $150°$, so $\triangle AEI$ is equilateral and its area is $\dfrac{(AI)^2\sqrt{3}}{4}$. Now, by the Law of Cosines, $AI^2 = 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot (-\dfrac{\sqrt{3}}{2})$, or $AI^2 = 52 + 24\sqrt{3}$. Therefore, $[AEI] = \dfrac{(52 + 24\sqrt{3})\sqrt{3}}{4} = \mathbf{18 + 13\sqrt{3}}$.

I-5. Consider the distance from the center of the circle $(4, 2)$ to the point $(7, 6)$. This distance is $\sqrt{(7-4)^2 + (6-2)^2} = 5$. However, the segment between those two points includes a radius of the circle, which has length 2. Therefore, the shortest distance between the circle and the point $(7, 6)$ is $5 - 2 = \mathbf{3}$.

I-6. Investigate the pattern: $337^2 = 113569$, $3337^2 = 11135569$, $33337^2 = 1111355569$, and so on, so that if there are n 3's in $333...337^2$, the resulting number has the form 1111...113555...569, where there are n 1's, a 3, $n - 1$ 5's, and a 69. To have 2000 digits, $n = 999$, and the digit sum is $999(1) + 3 + 998(5) + 6 + 9 = \mathbf{6007}$.

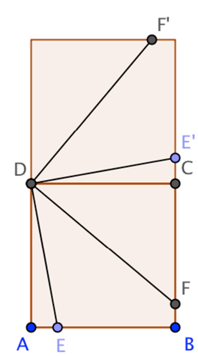
Incidentally, here is a proof that the pattern does indeed hold: $333...37 = 30(111...1) + 7$. Squaring, we obtain $900(111...1)^2$ (with n 1's) $+420(111...1)$ (with n 1's) $+49$, which is $(111...1)[900(111...1) + 420] + 49 = (111...1)[1000...0320] + 49$ (with $n - 1$ 0's). This is equal to 111...1000...0 (with n 1's and $n+2$ 0's) +333...300 (with n 3's) +222...20 (with n 2's) +49.

This gives the desired sum.

Or, as another of our authors has pointed out, you could compute 1011^2, 10011^2, and so on, and then divide each by 9 to obtain the desired numbers.

I-7. There are 9 one-digit numbers, $9 \cdot 10 = 90$ two-digit numbers, and $9 \cdot 10 \cdot 10 = 900$ three-digit numbers. The one-digit and two-digit numbers contribute 189 digits to the giant number. There are $2015 - 189 = 1826$ digits to be obtained from the three-digit numbers. Because $1826 = 3 \cdot 608 + 2$, we are looking for the second digit of the 609th three-digit number. The 609th number is 708, so the desired digit is **0**.

I-8. In order to create a segment with length $AE + CF$, one possibility is to rotate the square counterclockwise $90°$ about D. Let E' and F' be the images of E and F after the rotation. If we let $m\angle ADE = \alpha$, then $m\angle EDF = m\angle FDC = 45° - \alpha/2$, and $m\angle FDE' = m\angle FDC + m\angle CDE' = (45° - \alpha/2) + \alpha = 45° + \alpha/2$.



Looking at right triangle DCF, notice that $m\angle DFC = 45° + \alpha/2$. Thus, $\triangle DE'F$ is isosceles, which means $DE' = E'F = E'C + CF$. Because $DE' = DE$ and $E'C = EA$, $EA + CF = DE = \mathbf{20}$, and the length of $\overline{DA}$ was irrelevant to the solution of the problem.

Alternate Solution: Let $CE = x$ and $AE = a = \sqrt{20^2 - 15^2} = \sqrt{175} = 5\sqrt{7}$. In $\triangle DCF$, let $CF = b = 15\tan\theta$, where $\theta = m\angle CDF$. Using the Law of Cosines in $\triangle DCE$, $x^2 = 20^2 + 15^2 - 2 \cdot 20 \cdot 15 \cos 2\theta$. In $\triangle BCE$, $x^2 = (15 - a)^2 + 15^2$. Equating and cancelling, $(15-a)^2 = 400 - 600\cos 2\theta$. Substituting for a and reducing by 25, $(3-\sqrt{7})^2 = 16 - 24\cos 2\theta$, which implies $16 - 6\sqrt{7} = 16 - 24\cos 2\theta \rightarrow \cos 2\theta = \dfrac{\sqrt{7}}{4} = k$. From the double-angle formulas, we obtain $\sin^2\theta = \dfrac{1-k}{2}$ and $\cos^2\theta = \dfrac{1+k}{2}$, so $\tan^2\theta = \dfrac{1-k}{1+k} = \dfrac{(4-\sqrt{7})^2}{9}$, so $\tan\theta = \dfrac{4-\sqrt{7}}{3}$, and $b = 5(4-\sqrt{7}) \rightarrow a + b = 20$.

I-9. Look for a pattern. Let n_1 through n_6 be represented by a through f for convenience. Then, $n_7 = f - e + d - c + b - a$, $n_8 = (f - e + d - c + b - a) - f + e - d + c - b = -a$, and $n_9 = (-a) - (f - e + d - c + b - a) + f - e + d - c = -b$. Similarly, $n_{10} = -c$, $n_{11} = -d$, $n_{12} = -e$, $n_{13} = -f$, $n_{14} = -n_7$, and $n_{15} = n_1$, starting the pattern again. Since $2015 = 143 \cdot 14 + 13$, $n_{2015} = n_{13} = -n_6 = \mathbf{-15}$.
This is a "NYSML Classic". It is very much like question I7 from NYSML2005. Math never goes bad!

I-10. Let $f = \lfloor \sqrt{n} \rfloor$. Because $44^2 = 1936$ and $45^2 = 2025$, the value of f is at most 44. Because $f \le \sqrt{n} < f+1$, it is also true that $f^2 \le n < f^2+2f+1$. Because f is an integer, n is bounded between f^2 and $f^2 + 2f$. There are only three multiples of f in this range: f^2, $f^2 + f$, and $f^2 + 2f$. Thus, every positive integer f less than or equal to 44 generates 3 possible n values, except for 44 (because $44^2 + 2 \cdot 44 = 2024 > 2015$). The desired answer is $3 \cdot 44 - 1 = \mathbf{131}$.

Power Question 2015: Irregular Regular Polygons

The Regulars:

Recall that for any integer $n \geq 3$ there exists a **regular** polygon having n sides with all sides congruent and all internal angles congruent. For the purpose of this question, we assume that all sides have length 1, making this (convex) polygon R_n unique for each $n \geq 3$.

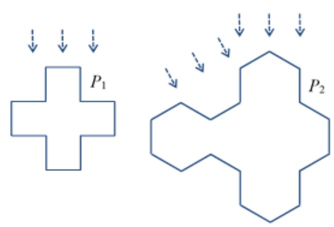
For any regular polygon R we define $\boldsymbol{\alpha(R)}$ as the degree-measure of any internal angle of R.

P-1. a. Compute the areas of R_3, R_4, R_6, and R_8. [4 pts]

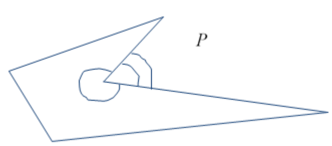
b. Provide an explicit formula (in terms of n) for $\alpha(R_n)$. [1 pt]

The Irregulars:

Consider the polygons P_1 and P_2 below. All of their sides have length 1, and for each of these polygons all of the non-reflex angles are congruent, but some of these angles are internal and some others are not. P_1 and P_2 are examples of **Irregular Regular Polygons** (IRPs).



Before we formally define an IRP, let's consider any polygon P. As usual, we do not allow self-intersecting polygons or polygons with overlapping vertices, but we do allow non-convex polygons.



Each vertex of P and two sides of P sharing this vertex form two angles. One of them is an internal angle of P and the other one is the corresponding explementary angle. Note that exactly one of these two angles is a reflex angle. Therefore, any polygon P with n vertices has n pairs of explementary angles, or $2n$ angles altogether – n reflex and n non-reflex ones.

A regular polygon could be defined as a convex polygon with all sides congruent and all non-reflex angles congruent. Now, we define an **IRP** as a non-convex polygon with all sides congruent and all non-reflex angles congruent. We continue to assume that all sides have length 1. Note that regular polygons R_n, because they are convex, are not IRPs.

For any regular polygon R we could define $\alpha(R)$ as the degree-measure of any non-reflex angle of R. Similarly, if P is an IRP, we define $\boldsymbol{\alpha(P)}$ as the degree-measure of any non-reflex angle of P, the IRP.

P-2. a. Compute the perimeters and areas of P_1 and P_2. [4 pts]

b. Compute the least possible radius of a disk which fully covers P_2. [1 pt]

P-3. a. Show that for every IRP P there exists an integer $n \geq 3$ such that $\alpha(P) = \alpha(R_n)$. [3 pts]

b. Show that every IRP has at least four internal non-reflex angles. [2 pts]

The Families:

For any integer $n \geq 3$ we define **IRP-$\boldsymbol{n}$** as the set (family) of all IRPs P such that $\alpha(P) = \alpha(R_n)$. The result of **P3-a** means that every IRP belongs to exactly one of these families. In the examples above, $P_1 \in$ IRP-4 and $P_2 \in$ IRP-6.

Hint: To solve some of the problems below, it might be useful to take a look at an IRP along a line parallel or perpendicular to one of the IRP's sides.

P-4. a. Show that the family IRP-3 is empty. [1 pt]

b. Draw two IRPs, $P_3 \in$ IRP-4 and $P_4 \in$ IRP-6, which have neither a line of symmetry nor a center of symmetry. [2 pts]

c. Draw two non-congruent IRPs, P_5 and P_6, having the same perimeters and the same areas. [2 pts]

P-5. a. Prove that every IRP from IRP-n has n more internal non-reflex angles than internal reflex angles. [2 pts]

b. Show that the perimeter of every IRP from IRP-n has the same parity as n. [1 pt]

c. Show that for every even integer $p \geq 10$, there exists an IRP from IRP-6 with perimeter p. [2 pts]

P-6. a. Show that the perimeter of every IRP from IRP-4 is at least 12. [1 pt]

b. Show that the perimeter of every IRP from IRP-n is at least $n + 2$. [1 pt]

c. Show that the perimeter of every IRP from IRP-n is at least $n + 4$. [3 pts]

P-7. a. Show that $2\alpha(R_5) + \alpha(R_{10}) = 360°$. [1 pt]

b. Draw an IRP from IRP-5 with perimeter 25. [2 pts]

c. Draw an IRP from IRP-5 with perimeter not equal to 25. [2 pts]

P-8. a. Prove that the perimeter of every IRP from IRP-5 is a multiple of 5. [3 pts]

b. Draw the unique IRP with the least possible perimeter. [2 pts]

P-9. a. Show that the family IRP-6 contains infinitely many different (non-congruent) IRPs. [1 pt]

b. Show that the family IRP-5 contains infinitely many different (non-congruent) IRPs. [2 pts]

c. Show that for every integer $n \geq 4$, the family IRP-n contains infinitely many different (non-congruent) IRPs. [2 pts]

P-10. Prove that there exists an IRP with a prime perimeter. [5 pts]

Solutions to 2015 Power Question

P-1. a. The area of R_3 is $\frac{\sqrt{3}}{4}$. The area of R_4 is 1. The area of R_6 is $\frac{3\sqrt{3}}{2}$. The area of R_8 is $2+2\sqrt{2}$.

b. Note that $\alpha(R_n) = 180^\circ \frac{n-2}{n} = 180^\circ \left(1 - \frac{2}{n}\right) = 180^\circ - \frac{360^\circ}{n}$.

P-2. a. Note that P_1 has perimeter 12 and area 5. Note also that P_2 has perimeter 18 and area $\frac{15\sqrt{3}}{2}$.

b. The least possible radius is $\sqrt{7}$.

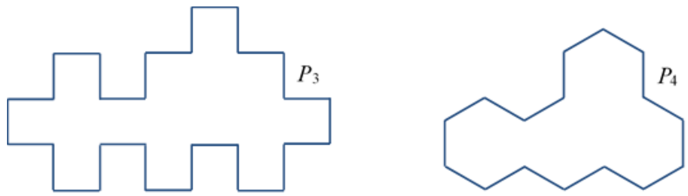
P-3. a. Let P be an arbitrary IRP, and $N = IR(P) + INR(P)$ be its number of vertices. As for any polygon with N vertices, the average degree-measure of an angle of P equals $\alpha(R_N) < 180^\circ$. One can pair as many reflex and non-reflex angles as possible (the average degree measure of an angle in each pair of explementary angles equals 180°). Any remaining angles after pairing must be non-reflex (having degree measure less than 180°) because otherwise the average degree measure of an angle of P would be at least 180°. Therefore, $INR(P) \geq IR(P)$, and it follows that there are $IR(P)$ pairs of explementary angles and $N - @IR(P)$ congruent non-reflex angles. Because the total degree measure of all angles of P equals $180^\circ(N-2)$, one obtains the following: $360^\circ IR(P) + \alpha(P)(N - 2IR(P)) = 180^\circ(N-2)$, which is equivalent to $\alpha(P)(N - 2IR(P)) = 180^\circ(N - 2IR(P) - 2)$, which is equivalent to $(180^\circ - \alpha(P))(N - 2IR(P)) = 360^\circ$. Because $0^\circ < \alpha(P) < 180^\circ$, this implies $N - 2IR(P) > 2$. Let $n = N - 2IR(P)$, where n is a natural number greater than or equal to 3. Then it follows that $\alpha(P) = 180^\circ \frac{n-2}{n}$. According to the answer from **P1-b**, $\alpha(R_n) = 180^\circ \frac{n-2}{n}$, and therefore $\alpha(P) = \alpha(R_n)$.

b. Let P be an arbitrary IRP, and $N = IR(P) + INR(P)$ be its number of vertices. As shown in **P3-a-Sol**, there exists $n \geq 3$ where $n = N - 2IR(P) = IR(P) + INR(P) - 2IR(P) = INR(P) - IR(P)$. Because the polygon P is non-convex, it follows that $IR(P) \geq 1$, and therefore $INR(P) = n + IR(P) \geq 4$.

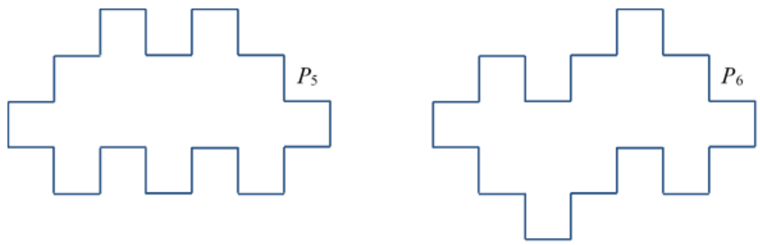
Alternatively, if $INR(P) \leq 2$, then $IR(P) \geq N - 2$, and the total degree measures of all reflex angles of P is greater than $180^\circ(N-2)$ (the total degree measure of all angles of P), which is a contradiction. If $INR(P) = 3$, then $IR(P) = N - 3 > 0$ (there is no such figure as a non-convex triangle!), and the total degree measure of all reflex angles of P is greater than $180^\circ(N-3)$. But from the result of **P3-a**, there is an integer $n \geq 3$ such that $\alpha(P) = \alpha(R_n)$, and the answer to **P1-b** implies $\alpha(R_n) \geq \alpha(R_3) = 60^\circ$, so the total degree measure of all non-reflex angles of P is at least $60^\circ(INR(P)) = 180^\circ$, and the total degree measure of all angles of P (which equals $180^\circ(N-2)$) is greater than $180^\circ(N-3) + 180^\circ = 180^\circ(N-2)$, which is a contradiction. Therefore, $INR(P) \geq 4$.

P-4. a. The answer to **P1-b** implies that $\alpha(R_3) = 60^\circ$. Start by drawing two segments of length 1 that make a 60° angle. The third segment of length 1 and the second segment should also make a 60° angle. So either the third segment closes off the figure, creating an equilateral triangle, or the third segment is parallel to the first segment. Now, to avoid a self-intersection, the fourth segment of length 1 must be parallel to the second segment, and this pattern continues; therefore, the figure is not closed.

b. Examples are shown below.



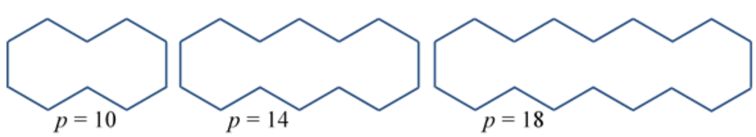
c. Start from P_3 and flip some U-shaped part of it consisting of three consecutive sides from "inside" to "outside", preserving the perimeter and increasing the area by 2. Flipping one part creates P_5, and flipping a congruent part in another place creates P_6.



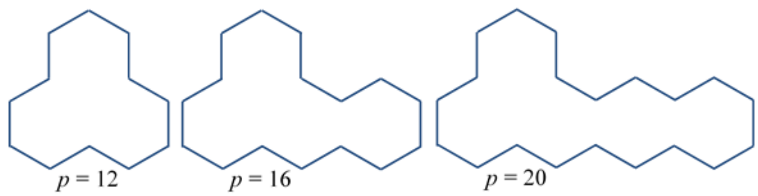
P-5. a. Let P be an arbitrary IRP from IRP-n and let $N = IR(P) + INR(P)$ be its number of vertices. Each non-reflex angle of P has degree measure $\alpha(P) = \alpha(R_n)$, and each reflex angle of P has degree measure $360° - \alpha(P)$. Because the total degree measure of all angles of P equals $180°(N - 2)$, it follows that $\alpha(P)INR(P) + (360° - \alpha(P))IR(P) = 180°(IR(P) + INR(P) - 2)$. The answer to **P1-b** implies that $\alpha(P) = 180° - \frac{360°}{n}$; therefore $360° - \alpha(P) = 180° + \frac{360°}{n}$, so simplify to obtain $\frac{-INR(P)}{n} + \frac{IR(P)}{n} = -1 \Leftrightarrow INR(P) - IR(P) = n$.

b. Let P be an arbitrary IRP from IRP-n and let $N = IR(P) + INR(P)$ be its perimeter. According to the result of **P5-a**, $INR(P) - IR(P) = n$, and therefore $N = IR(P) + IR(P) + n = 2IR(P) + n$ is of the same parity as n.

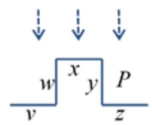
c. One can draw an IRP from IRP-6 with perimeter 10 by (externally) attaching two instances of R_6 to each other and removing their shared side (we can assume it was a vertical side). Attaching another instance of R_6 to the rightmost vertical side of the previous IRP and removing their shared side creates an IRP from IRP-6 with perimeter $10 + 4 = 14$. Attaching another instance of R_6 to the rightmost vertical side of the previous IRP and removing their shared side creates an IRP from IRP-6 with perimeter $14 + 4 = 18$. This pattern continues.



One can also draw an IRP from IRP-6 with perimeter 12 by (externally) attaching three instances of R_6 to each other and removing their three partially-shared sides (we can assume one of them was a vertical side). Attaching another instance of R_6 to the rightmost vertical side of the previous IRP and removing their shared side creates an IRP from IRP-6 with perimeter $12 + 4 = 16$. Attaching another instance of R_6 to the rightmost vertical side of the previous IRP and removing their shared side creates an IRP from IRP-6 with perimeter $16 + 4 = 20$. This pattern continues.



P-6. a. Let P be an arbitrary IRP from IRP-4. Assume that P has only horizontal and vertical sides. Let x be one of the top horizontal sides of P. Two neighboring sides, w and y, are vertical, and their neighboring sides, v and z, are horizontal. Neither of these two horizontal sides appear directly below x to avoid a self-intersection.



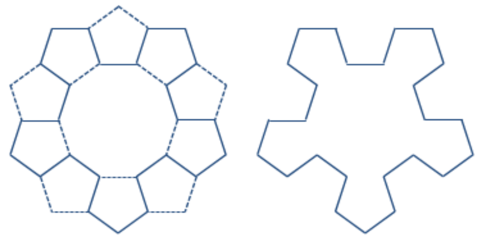
So if one looks at P from above along a vertical line, one will see at least three different horizontal sides (all of them differ from v, x, and z) not blocked by other sides. This means that P has at least six different horizontal sides. Similarly, P has at least six different vertical sides, and therefore its perimeter is at least 12.

b. Let P be an arbitrary IRP from IRP-n and let $N = IR(P) + INR(P)$ be its perimeter. From the result of **P5-a**, it follows that $INR(P) = n + IR(P)$, so $N = n + 2IR(P)$. Because polygon P is non-convex, it follows that $IR(P) \geq 1$ and therefore $N \geq n + 2$.

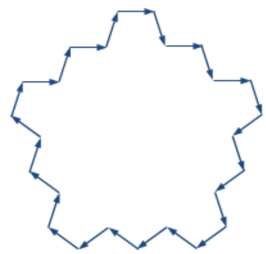
c. Assume there exists an IRP from IRP-n with perimeter $N = IR(P) + INR(P)$ where $N < x + 4$. The result of **P6-b** implies that $N \geq n + 2$, and the result of **P5-b** implies that N is of the same parity as n; therefore, N must be $n + 2$, and thus $INR(P) + IR(P) = n + 2$. The result of **P5-a** implies $INR(P) - IR(P) = n$, so it follows that $INR(P) = n + 1$ and $IR(P) = 1$. This means that P has $n + 1$ consecutive non-reflex angles with degree measure $\alpha(P) = \alpha(R_n)$. Therefore n consecutive sides of P form R_n, which is a contradiction.

P-7. a. The answer to **P1-b** implies that $\alpha(R_5) = 108°$ and $\alpha(R_{10}) = 144°$. Therefore, $2\alpha(R_5) + \alpha(R_{10}) = 360°$.

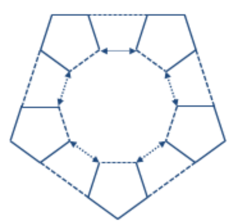
b. Draw R_{10} and then on each of its sides place an instance of R_5 (externally). The result of **P7-a** implies that each of these ten regular pentagons will share a side with two neighboring regular pentagons. Now it is straightforward to highlight some of their sides to get a required (flower-like) IRP from IRP-5 with perimeter 25 (at the right in the figure below).



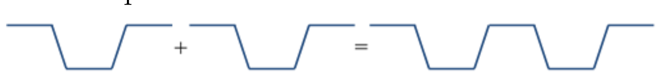
Alternatively, we can apply the vector-based method described in the solution to **P7-c** to get another IRP from IRP-5 with perimeter 25 (shown below).



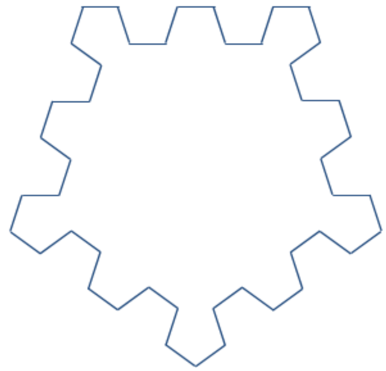
c. Consider the flower-like IRP, F, created in the solution to **P7-b**. One can obtain F by starting from a regular pentagon, A, with side length $2+a$, where a is the length of any diagonal of R_5. Then, in each of its five angles, place an instance of R_5. Consider the ten vertices of these smaller regular pentagons which are inside A and connect them, as shown in the diagram.



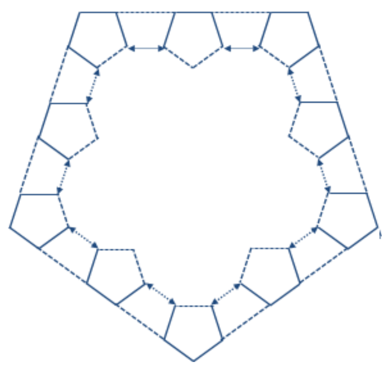
Finally, eliminate all sides of these smaller regular pentagons which are in the interior of A as well as all "central" parts of the sides of A with length a. Notice that F consists of five congruent parts, that each part consists of five sides of length 1, and that consecutive parts join each other at the same angle as neighboring sides of R_5 do. Now, take two such parts, glue them together as shown below to get a new longer part with length 9, and make a figure from five congruent longer parts joining them at the same angle as the one between the shorter parts of F.



The result is an IRP from IRP-5 (any two consecutive sides of the resulting polygon are consecutive sides of an instance of R_5) and its perimeter will be 45, which is different from 25.

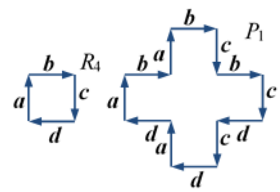


Alternate Solution: Start from a regular pentagon B with side length $3 + 2a$ where a is the length of any diagonal of R_5. Divide each side in the ratio $1 : a : 1 : a : 1$. Then, in each of its five angles, place an instance of R_5, and also place an instance of R_5 (inside B) on the central part of every side of B. Consider the twenty vertices of these ten smaller regular pentagons which are inside B and connect them, as shown.

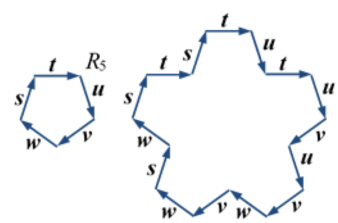


Eliminate all sides of these smaller regular pentagons which are in the interior of B as well as all parts of the sides of B with length a. The result is an IRP from IRP-5 (any two consecutive sides of the resulting polygon are consecutive sides of an instance of R_5) and its perimeter will be 45, which is different from 25.

Alternate Solution 2: Consider how one can obtain P_1 from R_5, and then apply a similar method to get some IRP from IRP-5 starting from R_5. To begin, draw R_4 and P_1 with horizontal and vertical sides, and consider their clockwise orientations (when traveling on a polygon in a clockwise direction, the polygon's interior would always be on the right). Every side becomes a vector. Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, and $\overrightarrow{d}$ be the side-vectors of R_4 (when traveling on it in a clockwise direction, the sides are in the order $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, and $\overrightarrow{d}$). Consider the order of the sides when traveling on P_1 in a clockwise direction: the sides are in the order $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$, $\overrightarrow{c}$, $\overrightarrow{d}$, $\overrightarrow{a}$, $\overrightarrow{d}$. One can obtain this sequence from the shorter sequence $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$ by simultaneously replacing each character with the corresponding 3-character sequence: $\overrightarrow{a}$ with $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{a}$; $\overrightarrow{b}$ with $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{b}$; $\overrightarrow{c}$ with $\overrightarrow{c}$, $\overrightarrow{d}$, $\overrightarrow{c}$; and $\overrightarrow{d}$ with $\overrightarrow{d}$, $\overrightarrow{a}$, $\overrightarrow{d}$. This gives insight into how to create an IRP from IRP-5.

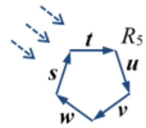


Draw R_5 and consider its clockwise orientation. Every side becomes a vector. Let $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ be the side-vectors of R_5 (when traveling on it in a clockwise direction, the sides are in the order $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$). Using the replacement method, we obtain the 15-character sequence $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$, $\overrightarrow{v}$, $\overrightarrow{w}$, $\overrightarrow{s}$, $\overrightarrow{w}$. Now, draw a figure which corresponds to this sequence. It will be a closed figure (the vectors $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ sum to a zero vector because R_5 is a closed figure, and therefore all fifteen vectors in the sequence will sum to a zero vector) which is not self-intersecting. The result is an IRP from IRP-5 (all of its sides will have length 1, and any two of its neighboring side-vectors appear as the neighboring side-vectors in R_5) and its perimeter will be 15, which is different from 25.



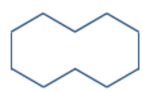
Note that if we start from the sequence $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ and simultaneously replace each character with a 5-character sequence similar to the one described above (for example, replacing $\overrightarrow{s}$ with $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{s}$), one obtains a 25-character sequence. The figure that corresponds to that sequence will be an IRP from IRP-5 with perimeter 25, representing an alternative solution to **P7-b**.

P-8. a. Let P be an arbitrary IRP from IRP-5, and N its perimeter. Draw R_5 and P with the corresponding sides parallel, and consider their clockwise orientations. Every side becomes a vector. Let $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ be the side-vectors of R_5 in clockwise order. Assume that the distinct side-vectors of P are also $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ (and if they have opposite directions we can reflect P over any of its sides). Let S, T, U, V, and W be the (integer) number of the side-vectors of P which are equal to $\overrightarrow{s}$, $\overrightarrow{t}$, $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ respectively. Then it follows that $S + T + U + V + W = N$, and the sum $S\overrightarrow{s} + T\overrightarrow{t} + U\overrightarrow{u} + V\overrightarrow{v} + W\overrightarrow{w}$ equals the zero vector. Consider P along a line parallel to its side-vector $\overrightarrow{w}$; that is, project all the vectors onto a line parallel to the side-vector $\overrightarrow{w}$.



It follows that $S\cos 18° + T\cos 54° - U\cos 54° - V\cos 18° = 0$. Using the fact that $\cos 54° = \sin 36° = 2\cos 18°\sin 18°$ and the fact that $\cos 18° \neq 0$, it follows that $(S-V) + 2(T-U)\sin 18° = 0$. If $T \neq U$, then it follows that $\sin 18°$ is a rational number, and this results in a contradiction. Therefore $T = U$. Similarly, it follows that $U = V$, $V = W$, and $W = S$, so $N = S+T+U+V+W = 5S$ is a multiple of 5.
Note that $\sin 18° = \cos 72°$, so if $\sin 18° = x$ for some x with $0 < x < 1$, then $\cos 36° = \cos(2\cdot 18°) = 1-2x^2$ and $\cos 72° = \cos(2\cdot 36°) = 2(1-2x^2)^2 - 1 = 8x^4 - 8x^2 + 1$, so one obtains $8x^4 - 8x^2 + 1 = x \leftrightarrow 8x^2(x^2-1) = x - 1$. Because $x \neq 1$, this implies $8x^2(x+1) = 1 \leftrightarrow (2x+1)(4x^2+2x-1) = 0$. Because $x > 0$, take the positive root of the quadratic equation $4x^2 + 2x - 1 = 0$, which implies $\sin 18° = \frac{\sqrt{5}-1}{4}$ and therefore $\sin 18°$ is irrational.

b. Let P be an arbitrary IRP from IRP-n, and let N be its perimeter. According to the result of **P4-a**, $n \geq 4$. If $n = 4$ then according to the result of **P6-a**, $N \geq 12$. If $n = 5$, then according to the results of **P5-b** and **P8-a**, N is an odd multiple of 5, and therefore $N \geq 15$. If $n \geq 7$, then according to the result of **P6-c**, $N \geq n + 4 \geq 11$. If $n = 6$, then according to the result of **P6-c**, $N \geq 10$. According to the result of **P5-c**, there exists an IRP from IRP-6 with perimeter 10. Therefore the least possible perimeter of an IRP is 10. The diagram contains an IRP with perimeter 10.

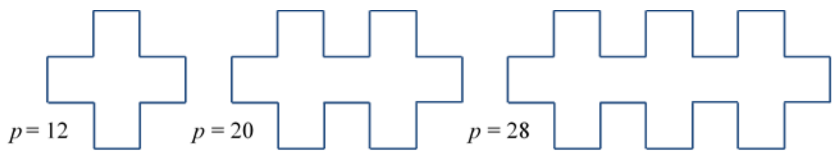


Such an IRP is indeed unique. As explained above, if an IRP P has perimeter 10, then it must be from IRP-6. According to the result of **P5-a**, $INR(P) - IR(P) = 6$, but $IR(P) + INR(P) = 10$, so it follows that $INR(P) = 8$ and $IR(P) = 2$. If P has five consecutive non-reflex angles with degree measure $\alpha(P) = \alpha(R_6)$, then six consecutive sides of P form R_n, which results in a contradiction. Therefore, eight non-reflex angles of P are split into two groups of four consecutive non-reflex angles, so when traveling on P in any direction the angles are in the following order (starting from a reflex one): reflex, non-reflex, non-reflex, non-reflex, non-reflex, reflex, non-reflex, non-reflex, non-reflex, non-reflex. According to the answer to **P1-b**, all non-reflex angles of P measure 120°, and therefore all reflex angles of P measure 240°. Because all sides of P have length 1, P is fully and uniquely (up to a congruency) defined by the above sequence of its ten consecutive angles.

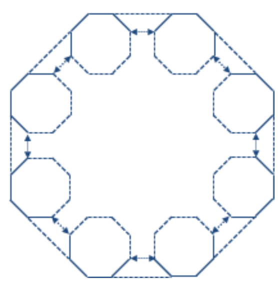
P-9. a. According to the result of **P5-c**, for every even integer $p \geq 10$ there exists an IRP from IRP-6 with perimeter p. Consider any IRP from IRP-6 with perimeter 10, any IRP from IRP-6 with perimeter 12, and so on. All of these IRPs are non-congruent because they have different perimeters, and there are infinitely many of them.

b. Any of the solutions of **P7-c** above could be generalized to provide a method of generating infinitely many IRPs from IRP-5 with different perimeters. All of these IRPs are non-congruent because they have different perimeters.
The generalization of the first (or the second) solution of **P7-c** allows for any positive integer p to create an IRP from IRP-5 with perimeter $5(4p + 1)$. For details, check the explanation and diagrams in the solution to **P9-c**.
The vector-based solution of **P7-c** is the easiest one to generalize. For any odd integer p greater than 2 we can create a $(5p)$-character sequence and draw the corresponding IRP from IRP-5 with perimeter $5p$.

c. Any of the solutions of **P7-c** requires the case where $n = 4$ to be considered separately. This could be done by starting from P_1 and keep extending it to the right similarly to what has been done in the solution to **P5-c**.

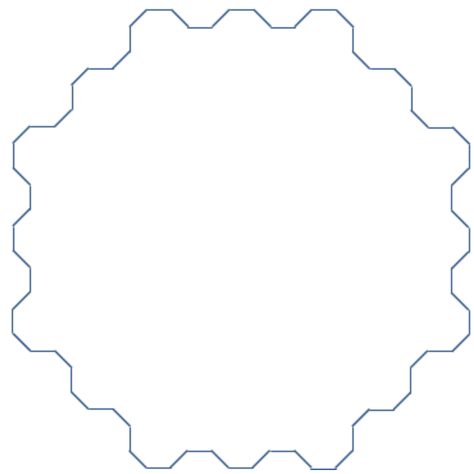


If $n \geq 5$, start from a regular n-gon A with side length $2 + a$, where a is the length of any diagonal of R_n which makes a trapezoid together with three consecutive sides of R_n. Then in each of its n angles place an instance of R_n. Consider the $2n$ vertices of these n smaller regular n-gons which are inside A and have one of their neighboring vertices on A, and connect them as shown in the diagram (note that in the diagram, $n = 8$).

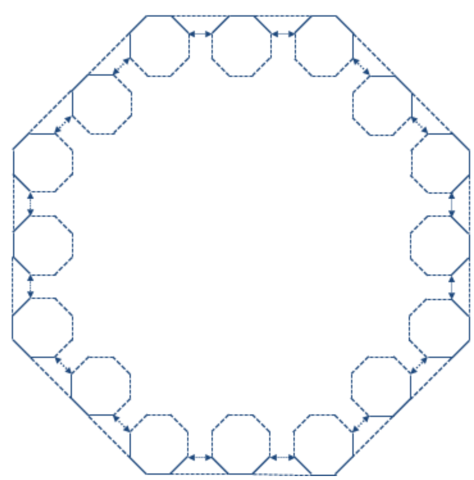


Finally, eliminate all sides of these smaller regular n-gons which are in the interior of A as well as all "central" parts of the sides of A with length a. One can see that the resulting IRP, F, consists of n congruent parts, that each part consists of 5 sides of length 1, and that consecutive parts join each other at the same angle as neighboring sides of R_n do. Now, for any positive integer p, take p such parts and glue them together as shown in the diagram to get a new longer part with length $pn - (p - 1)$, and make a figure from n congruent longer parts joining them at the same angle as the one between the shorter parts of F (note that in the diagram, $p = 2$).

We get an IRP from IRP-n (any two consecutive sides of the resulting polygon are consecutive sides of an instance of R_n), and its perimeter will be $n(pn - (p - 1))$.



Alternatively, choose a positive integer p and start from a regular n-gon, B, with side length $p+1+pa$, where a is the length of any diagonal of R_n which makes a trapezoid together with three consecutive sides of R_n, and divide each side in a ratio of $1:a:1:a:\ldots:a:1:a:1$. Then in each of its n angles place an instance of R_n, and also place an instance of R_n (inside B) on any non-corner part of length 1 of every side of B (in the diagram $n=8$ and $p=2$). Consider the $2pn$ vertices of these pn smaller regular n-gons which are inside B and have one of their neighboring vertices on B, and connect them as shown in the diagram.

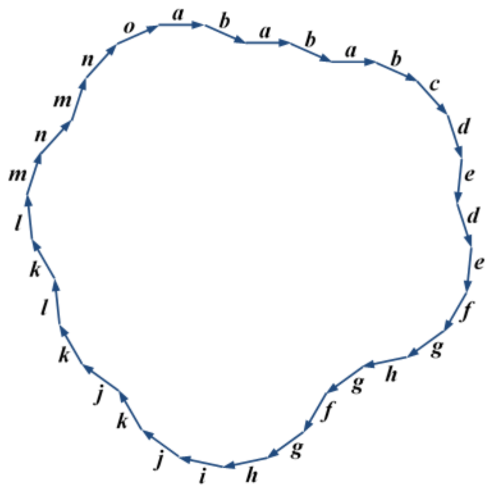


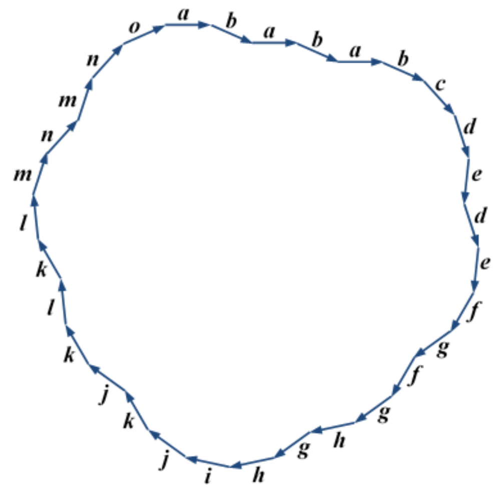
Finally, eliminate all sides of these smaller regular n-gons which are in the interior of B, as well as all parts of the sides of B with length a. We get an IRP from IRP-n (any two consecutive sides of the resulting polygon are consecutive sides of an instance of R_n), and its perimeter will be $n(pn–(p–1))$. The vector-based solution of **P7-c** is the easiest one to generalize. For any odd integer p greater than 2 we can create a (pn)-character sequence and draw the corresponding IRP from IRP-n with perimeter pn.

P-10. All the solutions of **P7-c** above produce IRPs with composite perimeters, but any of these solutions could be modified to produce an IRP with a prime perimeter. We will provide a modification of the vector-based solution.

In the vector-based solution of **P9-c** we started with an n-character sequence, took an odd integer p greater than 2, and replaced each character in the sequence with a p-character sequence to produce a (pn)-character sequence that corresponded to an IRP with perimeter pn. To produce an IRP with a prime perimeter using a similar approach, we cannot replace all characters in the original n-character sequence. But replacing only some characters may lead to a character sequence which does not correspond to a polygon (we may end up with a non-closed figure, i.e. the end of the last vector may not coincide with the beginning of the first vector). But if the vectors we are going to add to the sequence sum up to a zero vector, the new sequence will produce a polygon (formally, we will only avoid the non-closing issue; a self-intersection may also become an issue in some cases, but in our specific case below, it will not). Since we always add an even number of characters, we need to use an odd n. To be able to easily add some, but not all, available vectors that sum up to a zero vector, we can take an odd composite n. If $n = 9$, we will always add a number of vectors which is a multiple of 3, so the number of characters in the resulting sequence will also be a multiple of 3. The next possible value of n is 15 (a product of the two smallest distinct odd primes).

Everything in the above paragraph was just a hint to the actual solution. Let's draw R_{15} and consider its clockwise orientation. Every side becomes a vector. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, and $\vec{o}$ be the side-vectors of R_{15} (when traveling on it in a clockwise direction, the sides are in the order $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, and $\vec{o}$). These 15 vectors sum up to a zero vector since R_{15} is a closed figure. Note that due to symmetry vectors $\vec{a}$, $\vec{f}$, and $\vec{k}$ could serve as side-vectors of R_3 (and therefore they sum up to a zero vector), and vectors $\vec{a}$, $\vec{d}$, $\vec{g}$, $\vec{j}$, and $\vec{m}$ could serve as side-vectors of R_5 (and therefore they sum up to a zero vector). The same is true for vectors $\vec{b}$, $\vec{g}$, $\vec{\ell}$ and $\vec{b}$, $\vec{e}$, $\vec{h}$, $\vec{k}$, $\vec{n}$, respectively. So we start from the 15-character sequence $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, $\vec{o}$ and simultaneously replace each of the characters $\vec{a}$, $\vec{f}$, and $\vec{k}$ with the corresponding 3-character sequences: $\vec{a}$ with $\vec{a}$, $\vec{b}$, $\vec{a}$; $\vec{f}$ with $\vec{f}$, $\vec{g}$, $\vec{f}$; and $\vec{k}$ with $\vec{k}$, $\vec{\ell}$, $\vec{k}$. After replacing we obtain the 21-character sequence $\vec{a}$, $\vec{b}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, $\vec{o}$ which corresponds to an IRP from IRP-15 with perimeter 21. Note that these 21 vectors sum up to a zero vector. Now we simultaneously replace one of the characters $\vec{a}$, character $\vec{d}$, one of the characters $\vec{g}$, and characters $\vec{j}$ and $\vec{m}$ with the corresponding 3-character sequences: $\vec{a}$ with $\vec{a}$, $\vec{b}$, $\vec{a}$; $\vec{d}$ with $\vec{d}$, $\vec{e}$, $\vec{d}$; $\vec{g}$ with $\vec{g}$, $\vec{h}$, $\vec{g}$; $\vec{j}$ with $\vec{j}$, $\vec{k}$, $\vec{j}$; and $\vec{m}$ with $\vec{m}$, $\vec{n}$, $\vec{m}$. After replacing we obtain either the 31-character sequence $\vec{a}$, $\vec{b}$, $\vec{a}$, $\vec{b}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{g}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, $\vec{m}$, $\vec{n}$, $\vec{o}$ or the 31-character sequence $\vec{a}$, $\vec{b}$, $\vec{a}$, $\vec{b}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{d}$, $\vec{e}$, $\vec{f}$, $\vec{g}$, $\vec{f}$, $\vec{g}$, $\vec{h}$, $\vec{g}$, $\vec{h}$, $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{j}$, $\vec{k}$, $\vec{\ell}$, $\vec{k}$, $\vec{\ell}$, $\vec{m}$, $\vec{n}$, $\vec{m}$, $\vec{n}$, $\vec{o}$. Each of them corresponds to an IRP from IRP-15 with the prime perimeter 31 (see the diagram). Note that these 31 vectors sum up to a zero vector.


a b a b a b c d e d e f g h g f g h i j k j k l k l m n m n o


a b a b a b c d e d e f g f g h g h i j k j k l k l m n m n o

2015 Relay Problems

R1-1. Compute the two-digit number that is equal to one more than three times the sum of its digits.

R1-2. *Let N be the two-digit prime you will receive.* The four-digit number $Y = \underline{2}\,\underline{0}\,\underline{P}\,\underline{Q}$ is divisible by N. Compute the number of distinct possible values of Y.

R1-3. *Let N be the number you will receive.* The solutions of $x^3 - 4x^2 + 6x - N = 0$ are p, q, and r. Compute the value of $(p+q)(p+r)(q+r)$.

R2-1. Compute the whole number that is 29760 less than its cube.

R2-2. *Let N be the number you will receive.* Sam has N coins, each of which is either a nickel or a quarter. The total value of the N coins is \$5.75. Compute the number of nickels Sam has.

R2-3. *Let N be the number you will receive.* If Susie stands on the up escalator and rides it from the first floor to the second floor, the trip takes 12 seconds. If instead Susie walks up a parallel staircase from the first floor to the second floor, the trip takes N seconds. If Susie walks up the up escalator from the first floor to the second floor, the trip takes $\dfrac{A}{B}$ seconds, where A and B are positive integers whose greatest common factor is 1. Compute $A + B$.

2015 Relay Answers

R1-1. 13

R1-2. 8

R1-3. 16

R2-1. 31

R2-2. 10

R2-3. 71

2015 Relay Solutions

R1-1. Let the two-digit number $\underline{T}\,\underline{U}$ be expressed as $10T + U$. Then, we seek to solve $10T + U = 1 + 3(T + U) \Rightarrow 7T = 2U + 1$. If $T = 1$, then $U = \dfrac{7-1}{2} = 3$. If $T > 1$, then U is not a digit. Therefore, the only two-digit number satisfying the criterion is **13**.

R1-2. Since $2015 = 5 \cdot 13 \cdot 31$, it is true that 2015 is one such number, and therefore so is $2015 - 13 = 2002$. Adding 13 until the numbers exceed 2099 yields 2028, 2041, 2054, 2067, 2080, and 2093, for a total of **8** possible values of Y.

R1-3. Note that the sum of the roots is $p + q + r = 4$, so $p + q = 4 - r$ and $p + r = 4 - q$ and $q + r = 4 - p$. Note also that the cubic equation can be written as $(x - p)(x - q)(x - r) = 0$. Therefore, the desired quantity is $f(4)$, which is $64 - 64 + 24 - 8 = \mathbf{16}$.

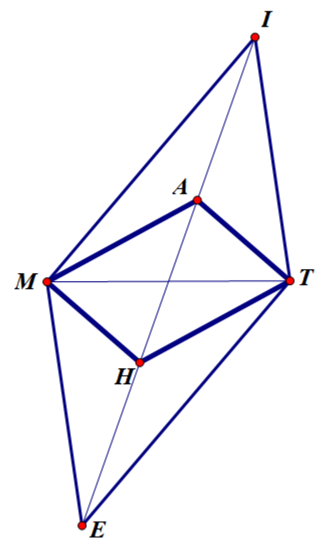
R2-1. The equation to solve is $x^3 - x = 29760 \to x(x+1)(x-1) = 29760$. Factoring yields $x(x+1)(x-1) = 2^6 \cdot 3 \cdot 5 \cdot 31 = 30 \cdot 31 \cdot 32$. Therefore, $x = \mathbf{31}$.

R2-2. Let x be the number of nickels. Then $5x + 25(N - x) = 575 \to x = \dfrac{5N - 115}{4}$. Substituting, $x = \mathbf{10}$.

R2-3. Let the distance from the first floor to the second floor be d. Then, the escalator moves at a rate of $\frac{d}{12}$ while Susie walks at a rate of $\frac{d}{N}$. If Susie walks up the up escalator, this trip takes time given by $\dfrac{d}{\frac{d}{12} + \frac{d}{N}} = \dfrac{12N}{N + 12}$. Substituting $N = 10$ and reducing, we obtain $\dfrac{60}{11}$. The desired sum is $60 + 11 = \mathbf{71}$.

2015 Tiebreaker Problems

TB-1. In parallelogram $MATH$, $MA = 11$ and $AT = 9$. In parallelogram $TIME$, $TI = 13$ and $MI = 17$. Vertices A and H of $MATH$ trisect diagonal $\overline{IE}$ of $TIME$. Compute the length IE.



TB-2. The three-digit octal (base-8) number $N = \underline{A}\,\underline{B}\,\underline{C}$ is 5 times the two-digit octal number $\underline{A}\,\underline{C}$. Compute the greatest possible value of N, giving your answer in base 8.

2015 Tiebreaker Answers

TB-1. 24

TB-2. 106

2015 Tiebreaker Solutions

TB-1. Apply Stewart's Theorem twice to triangle MEI, once using $\overline{MH}$ as the cevian, and once using $\overline{MA}$ as the cevian. Let $x = EH = HA = AI$. By Stewart's Theorem, we obtain

$$13^2(2x) + 17^2(x) = 3x(9^2 + 2x^2)$$
$$13^2(x) + 17^2(2x) = 3x(11^2 + 2x^2).$$

Adding these equations, we obtain $3x(13^2 + 17^2) = 3x(9^2 + 11^2 + 4x^2)$. This can be solved to obtain $x = 8$, so $3x = IE = \mathbf{24}$.

TB-2. We have that $64A+8B+C = 5(8A+C)$, which implies $24A+8B-4C = 0 \rightarrow C = 2(3A+B)$. Since the value of C must be no greater than 7 and C must be even, the greatest value of C is 6. That makes $3A + B = 3$, and thus $A = 1$ and $B = 0$. The greatest solution has $N = \mathbf{106}$.

2016 Contest at Penfield High School (Monroe)

Contents

2016 Team Problems

T-1. The base-10 number 2016 can be expressed using base 3 as 2202200, four 2's and three 0's. There are two base-10 numbers A and B that are greater than 2016 and whose base-3 representations use four 2's and three 0's. Given that $A < B$, compute (A, B).

T-2. Suppose that, for a sequence $\{a_n\}$, $a_1 = 435$, $a_2 = 167$, and $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$. Compute $a_{20} - a_{16} + a_{2016}$.

T-3. Compute all n such that $n + s(n) = 2016$, where $s(n)$ denotes the sum of the digits of n.

T-4. Let S be the collection of 121 (not necessarily distinct) parabolas of the form $f(x) = (x - a)(x - b)$ where $a, b \in \{0, 1, 2, \ldots, 10\}$. The average of the y-coordinates of the vertices of the 121 parabolas is t. Compute t.

T-5. To reach the Forbidden City, one must climb a lot of identical steps. Yodely Guy always starts on the first step. Afterwards...
If Yodely Guy skips every other step, he will reach the top step on his last stride.
If Yodely Guy skips two steps on each stride, he will reach the top step on his last stride.
If Yodely Guy skips four steps on each stride, he will reach the top step on his last stride.
Given that there are more than 500 steps, compute the minimum number of steps in the staircase to the Forbidden City.

T-6. Suppose that $f(x)$ is defined for all integers $x \geq 0$, and that $f(m+n) = f(m)+f(n)-2\cdot f(m\cdot n)$ for all integers m and n. Given that $f(9) = 224$, compute $f(2015) + f(2016) + f(2017)$.

T-7. Compute the greatest multiple of 11 all of whose digits are distinct.

T-8. A sphere of radius 1 is inscribed in a cube with edge-length 2. Eight smaller congruent spheres of radius r are placed, one in each corner of the cube, so that each smaller sphere is tangent to three faces of the cube and to the larger sphere. Compute r.

T-9. For all real values of x and y, compute the minimum value of the expression

$$2|x - 5| + 11|10 - y| + |11y - 2x + 2000|.$$

T-10. In $\triangle ABC$, $AB = 10$, $BC = 21$, and $CA = 17$. Point D is on $\overline{BC}$ such that $CD = 12$. Circle O passes through A, C, and D. Given that E is on circle O such that $\overline{CE}$ is parallel to $\overline{AB}$, compute CE.

2016 Team Answers

T-1. $(2124, 2160)$

T-2. 870

T-3. 1989 and 2007 *[must have both]*

T-4. -5

T-5. 511

T-6. 448

T-7. 9876524130

T-8. $2 - \sqrt{3}$

T-9. 2100

T-10. $\dfrac{37}{10}$ or $3\dfrac{7}{10}$ or 3.7

2016 Team Solutions

T-1. The two numbers can be found by "sliding" the first 0 "down" the number 2202200 to create 2220200 and 2222000. Converting these gives $A = 2124$ and $B = 2160$, so the desired ordered pair is $\boxed{\mathbf{(2124, 2160)}}$.

T-2. There is a pattern: $a_3 = a_2 - a_1$, $a_4 = a_3 - a_2 = -a_1$, $a_5 = a_4 - a_3 = -a_2$, $a_6 = a_5 - a_4 = a_1 - a_2 = -a_3$, and $a_7 = a_1$. Thus, the sequence repeats with period 6. So, $a_{20} - a_{16} + a_{2016} = a_2 + a_1 + (a_1 - a_2) = 2a_1 = 2 \cdot 435 = \mathbf{870}$.

T-3. Because $n \equiv s(n) \pmod 9$, it is true that $n \equiv 0 \pmod 9$. The greatest n which is less than 2016 and also equivalent to 0 mod 9 is 2007, which satisfies the equation. The greatest n which is less than 2007 and also equivalent to 0 mod 9 is 1998, which does not satisfy the equation. Next, note that the next smallest such integer is 1989, which does satisfy the given equation. No other integer may satisfy the equation because $s(n) \leq 28$ for four-digit integers less than 2000 and that means that $n \geq 1988$. Thus, n could be **1989** or **2007**.

T-4. The vertices are at:

	0	1	2	3	$\cdots$	10	x-coord	y-coord
0	$(0,0)$	$(\frac{1}{2}, \frac{-1}{4})$	$(1,-1)$	$(\frac{3}{2}, \frac{-9}{4})$	$\cdots$	$(5,-25)$	$\frac{0}{2}$ to $\frac{10}{2}$	$\frac{0^2}{4}$ to $\frac{-10^2}{4}$
1	$(\frac{1}{2}, \frac{-1}{4})$	$(1,0)$	$(\frac{3}{2}, \frac{-1}{4})$	$(2,-1)$	$\cdots$	$(\frac{11}{2}, \frac{-81}{4})$	$\frac{1}{2}$ to $\frac{11}{2}$	$\frac{-1^2}{4}$ to $\frac{0^2}{4}$ to $\frac{-9^2}{4}$
2	$(1,-1)$	$(\frac{3}{2}, \frac{-1}{4})$	$(2,0)$	$(\frac{5}{2}, \frac{-1}{4})$	$\cdots$	$(6,-16)$	$\frac{2}{2}$ to $\frac{12}{2}$	$\frac{-2^2}{4}$ to $\frac{0^2}{4}$ to $\frac{-8^2}{4}$
3	$(\frac{3}{2}, \frac{-9}{4})$	$(2,-1)$	$(\frac{5}{2}, \frac{-1}{4})$	$(3,0)$	$\cdots$	$(\frac{13}{2}, \frac{-49}{4})$	$\frac{3}{2}$ to $\frac{13}{2}$	$\frac{-3^2}{4}$ to $\frac{0^2}{4}$ to $\frac{-7^2}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
10	$(5,-25)$	$(\frac{11}{2}, \frac{-81}{4})$	$(6,-16)$	$(\frac{13}{2}, \frac{-49}{4})$	$\cdots$	$(10,0)$	$\frac{10}{2}$ to $\frac{20}{2}$	$\frac{-10^2}{4}$ to $\frac{0^2}{4}$

The value of t is $\frac{1}{121}\left(\frac{-1}{4}\left(2(10)^2 + 4(9^2) + 6(8^2) + 8(7^2) + \cdots + 20(1^2) + 11(0^2)\right)\right)$, or

$\frac{-1}{484}(200 + 324 + 384 + 392 + 360 + 300 + 224 + 144 + 72 + 20 + 0) = \mathbf{-5}$.

T-5. The number of steps is one more than a multiple of 2, 3, and 5. The least common multiple of 2, 3, and 5 is 30. Thus, the least multiple of 30 greater than 500 is 510, so our answer is **511**.

T-6. We first show that $f(2 \cdot k) = 0$ for all natural k, and then we show that $f(2 \cdot k + 1) = f(1)$ for all natural k.
Note that if $m = n = 0$, then $f(0) = f(0) + f(0) - 2f(0) = 0$, and if $m = n = 1$, then $f(2) = f(1) + f(1) - 2f(1) = 0$ also.
Now proceed by induction. Let $f(j) = 0$ for even j less than or equal to $2k$ and let $f(j) = f(1)$

for odd j between 1 and $2k-1$. Then, inductively, $f(2k+1)$ equals $f(1)+f(2k)-2f(2k)$. By the inductive hypothesis, $f(2k+1) = f(1)+0-0 = f(1)$. Note that

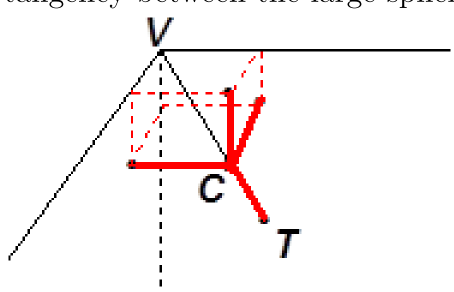
$$f(2k+2) = f(1) + f(2k+1) - 2f(2k+1) = f(1) + f(1) - 2f(1) = 0.$$

Thus the sequence is $f(1)$, 0, $f(1)$, 0, ... The answer is $f(2015) + f(2016) + f(2017) = 224 + 0 + 224 = \mathbf{448}$.

This is a "NYSML Classic". It is very much like question T9 from NYSML1986. For Met fans, this is not the only part of 1986 that needs to be revisited!

T-7. Assume that the number contains all ten digits and that it is of the form $98765ABCDE$. This number is divisible by 11 if and only if $(9+7+5+B+D)-(8+6+A+C+E)$ is a multiple of 11. This implies that $7+(B+D-(A+C+E)) = 11k$ for some integer k. This is only possible if $k=1$ and this requires $B+D = 4+A+C+E$. This is possible if $\{B, D\} = \{4, 3\}$ and $\{A, C, E\} = \{0, 1, 2\}$. Thus, the greatest multiple of 11 with distinct digits is **9876524130**.

T-8. The space diagonal of the cube has length $2\sqrt{3}$. Each space diagonal will pass through the center of the large sphere and two of the smaller spheres (as well as the points of tangency between the spheres. Let C denote the center of one of the small spheres and T the point of tangency between the large sphere and one of the smaller spheres.



Notice that $\overline{VC}$ is a space diagonal of a cube with edge r and $\overline{TC}$ is a radius of a small sphere. Thus, $2r + 2r\sqrt{3} + 2 = 2\sqrt{3}$. We solve to find $r = \dfrac{2\sqrt{3}-2}{2\sqrt{3}+2} = \mathbf{2-\sqrt{3}}$.

T-9. Using the inequality $|A|+|B| \geq |A+B|$, where $A = 2(x-5)$ and $B = 11(10-y)$, obtain $2|x-5| + 11|10-y| \geq |2x-11y+100|$. Now let $A = 2x-11y+100$ and $B = 11y-2x+2000$ and apply the inequality again to obtain $|2x-11y+100| + |11y-2x+2000| \geq 2100$ or $|2x-11y+100| \geq 2100 - |11y-2x+2000|$. Applying the Transitive Property and moving the rightmost term to the other side, we have that the given expression is at least **2100**. Notice that this can be achieved for many ordered pairs (x, y); for example, $(x, y) = \left(5, -\dfrac{1990}{11}\right)$ satisfies this.

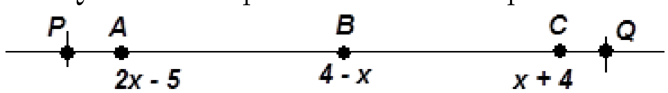
T-10. Extend $\overline{AD}$ and $\overline{EC}$, and let F be their intersection. Then $\triangle FCD \sim \triangle ABD$. Use the Law of Cosines to find that $\cos B = \dfrac{3}{5}$ and apply the Law of Cosines to $\triangle ABD$ to obtain $AD = \sqrt{73}$. Using similar triangles, $FD = \dfrac{4\sqrt{73}}{3}$ and $FC = \dfrac{40}{3}$. Now, use the Power of a Point Theorem on secants $\overline{FA}$ and $\overline{FE}$ to obtain $\dfrac{4\sqrt{73}}{3}\left(\dfrac{4\sqrt{73}}{3} + \sqrt{73}\right) = \dfrac{40}{3}\left(\dfrac{40}{3} + CE\right)$. This solves to obtain $CE = \mathbf{\dfrac{37}{10}}$.

2016 Individual Problems

I-1. Consider sets of numbers $\{20, 16, m\}$ for various values of m. If the median of the three numbers is equal to the mean of the three numbers, compute the sum of all possible values of m.

I-2. A five-digit natural number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}\,\underline{E}$ is called *downup* if $A > B > C$ and $C < D < E$. For example, the numbers 96368 and 32014 are downup. Compute the number of downup five-digit natural numbers.

I-3. The diagram shows a number line. On the number line, B is the average of A and C. For some real x, $A = 2x - 5$, $B = 4 - x$, and $C = x + 4$. P and Q are integers such that $P < A$ and $Q > C$. Compute the minimum possible distance PQ.



I-4. Compute all real values of x such that $x^2 + 3x - 26 = \sqrt{16x^2 + 48x - 80}$.

I-5. Compute the sum of the seven (not necessarily distinct) prime factors of $8! - 5!$.

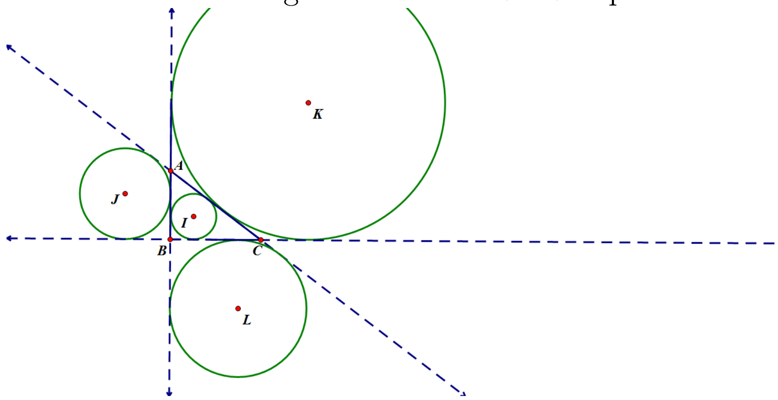
I-6. In $\triangle ABC$, $\sin^2 A + \sin^2 B = \sin^2 C + \sin A \sin B \sin C$. Compute $\sin C$ in the form $\dfrac{x}{y}$ where x is real and y is a positive integer.

I-7. The area enclosed by the graph of $|ax| + |2ay| = 6$ is 2016. Compute the positive value of a.

I-8. For some positive integer values of n and k, $2\dbinom{n}{k} = \dbinom{n}{k+1}$ and $3\dbinom{n}{k} = \dbinom{n}{k+2}$. Compute $n + k$.

I-9. The roots of $x^2 + 30x + 40 = 0$ are s and t with $s < t$. Compute $(s+20)(s+16) + (t+20)(t+16)$.

I-10. Triangle ABC has side lengths $AB = 3$, $BC = 4$, and $AC = 5$. A circle centered at I is tangent to all three sides of $\triangle ABC$. Circles centered at J, K, and L are each tangent to the three lines containing sides of $\triangle ABC$. Compute the sum of the areas of the four circles.



2016 Individual Answers

I-1. 54

I-2. 2892

I-3. 8

I-4. -9 and 6 *[must have both]*

I-5. 86

I-6. $\dfrac{2\sqrt{5}}{5}$

I-7. $\dfrac{\sqrt{14}}{28}$

I-8. 18

I-9. 380

I-10. 50π

2016 Individual Solutions

I-1. The mean is equal to $\dfrac{36+m}{3}$. If the mean is equal to 16, then solve $\dfrac{36+m}{3} = 16 \to m = 12$. If the mean is equal to 20, then solve $\dfrac{36+m}{3} = 20 \to m = 24$. If the mean is equal to m, then solve $\dfrac{36+m}{3} = m \to m = 18$. The sum of all possible values of m is $12 + 24 + 18 = \mathbf{54}$.

I-2. The values of C range from 0 to 7. Once C is chosen, there are $9 - C$ digits greater than C, of which two must be chosen to be A and B and of which two must be chosen to be D and E. Thus, there are $\left(\dbinom{9-C}{2}\right)^2$ downup numbers for each value of C. Summing these from $C = 0$ to $C = 7$, the desired total is $36^2 + 28^2 + 21^2 + 15^2 + 10^2 + 6^2 + 3^1 + 1^2 = \mathbf{2892}$.

I-3. Using the definition of average, $4 - x = \dfrac{(2x-5)+(x+4)}{2} \to 8 - 2x = 3x - 1$, so $x = \dfrac{9}{5}$. Now, $(A, B, C) = (2x - 5, 4 - x, x + 4) = \left(-\dfrac{7}{5}, \dfrac{11}{5}, \dfrac{29}{5}\right)$. The minimum distance PQ will occur when P is the greatest integer less than A and when Q is the least integer greater than C. Thus, $P = -2$ and $Q = 6$, for a minimum $PQ = \mathbf{8}$.

I-4. Let $y = \sqrt{x^2 + 3x - 5}$. Then, the given equation can be expressed as $y^2 - 21 = 4y$, and this is equivalent to $y^2 - 4y - 21 = 0$. Factoring and solving, we obtain $y = -3$ (which is rejected) or $y = 7$. Now, solve $x^2 + 3x - 5 = 49$ to obtain $x = \mathbf{-9}$ or $x = \mathbf{6}$.

This is a "NYSML Classic". It is very much like question I7 from NYSML2006. Math never goes bad!

I-5. Remove the greatest common factor of 5! to obtain $5!(8 \cdot 7 \cdot 6 - 1) = 335(5 \cdot 2^2 \cdot 3 \cdot 2)$, or $67 \cdot 5^2 \cdot 3 \cdot 2^3$, so the sum is $67 + 5(2) + 3 + 2(3) = \mathbf{86}$.

I-6. The sides and sines of angles in $\triangle ABC$ obey the Law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, which means that $\sin A = ka$ and $\sin B = kb$ and $\sin C = kc$ for some k (which is incidentally $\dfrac{1}{2R}$ where R is the circumradius of the triangle). Substitute to obtain the equivalent equation $k^2a^2 + k^2b^2 = k^2c^2 + k^2ab\sin C$, which implies $a^2 + b^2 = c^2 + ab\sin C$. However, the Law of Cosines also holds true for this triangle, so $a^2 + b^2 = c^2 + 2ab\cos C$. Thus, $2\cos C = \sin C$.

Finally, use $\sin^2 C + \cos^2 C = 1$ to obtain $\sin C = \mathbf{\frac{2\sqrt{5}}{5}}$. *Note: Because angle C is an angle in a triangle, the value of* $\sin C$ *must be non-negative.*

I-7. The graph is a rhombus with vertices $(0, 3/a)$, $(6/a, 0)$, $(0, -3/a)$, and $(-6/a, 0)$. The area of a rhombus is half the product of the lengths of its diagonals, so the area is $\frac{1}{2} \cdot \frac{12}{a} \cdot \frac{6}{a} = 2016$. This is equivalent to $\frac{36}{a^2} = 2016$, so $a^2 = \frac{1}{56} \to a = \frac{1}{\sqrt{56}} = \mathbf{\frac{\sqrt{14}}{28}}$.

I-8. The equation $2\binom{n}{k} = \binom{n}{k+1}$ implies $\frac{n!}{k!(n-k)!} = \frac{n!}{2(k+1)!(n-k-1)!}$. Expanding the denominators yields $n - k = 2(k + 1)$, so $n = 3k + 2$. The equation $3\binom{n}{k} = \binom{n}{k+2}$ implies $\frac{n!}{k!(n-k)!} = \frac{n!}{3(k+2)!(n-k-2)!}$. Expanding the denominators yields $(n - k)(n - k - 1) = 3(k+2)(k+1)$. Substituting, $(2k+2)(2k+1) = 3(k+2)(k+1) \to 2(2k+1) = 3(k+2) \to k = 4$, so $n = 14$, and $n + k = \mathbf{18}$.

Alternate Solution: Take note that $\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = 1 : 2 : 3$. In Pascal's Triangle, a 1:2 ratio appears in every third row with $\binom{n}{k+1} = \binom{3x-1}{x}$ starting in Row 2. Similarly, a 2:3 ratio appears in every fifth row with $\binom{n}{k+1} = \binom{5y-1}{2y-1}$ starting in Row 4. So, $n \equiv 14 \pmod{15}$. This sets up the system of equations where $3x-1 = 5y-1$ and $x = 2y-1$. Thus, $y = 3$ and $x = 5$. The relevant elements are $\binom{14}{4} : \binom{14}{5} : \binom{14}{6} = 1001 : 2002 : 3003$. The value of $n + k = 14 + 4 = \mathbf{18}$.

I-9. For a given root r of $x^2 + 30x + 40 = 0$, note that

$$(r + 20)(r + 16) = r^2 + 36r + 320 = (r^2 + 30r + 40) + 6r + 280,$$

or simply, $(r+20)(r+16) = 6r+280$. Thus, $(s+20)(s+16)+(t+20)(t+16) = 6s+280+6t+280$, which can be rearranged to be $6(s+t)+560$. Now, use Vieta's formulas to obtain $s+t = -30$, so $(s + 20)(s + 16) + (t + 20)(t + 16) = 6(-30) + 560 = \mathbf{380}$.

I-10. Let the inradius of $\triangle ABC$ be r_I, and let the radii of the three excircles be r_J, r_K, and r_L. The area of $\triangle ABC$, denoted $[ABC]$, is equal to $s \cdot r_I$, where $s = \frac{3+4+5}{2} = 6$. Thus,

$\frac{1}{2} \cdot 3 \cdot 4 = 6 \cdot r_I$ implies $r_I = 1$. Also, the area of a triangle is also equal to the product of the radius of the excircle that is tangent to the extensions of the sides containing that vertex and the difference between the semiperimeter of the triangle and the length of the side to which the excircle is tangent. Thus, $6 = (6-3)r_J \to r_J = 2$, $6 = (6-5)r_K \to r_K = 6$, and $6 = (6-4)r_L \to r_L = 3$. The total area of the four circles is $\pi\left(1^2 + 2^2 + 3^2 + 6^2\right)$, or $50\boldsymbol{\pi}$.

Power Question 2016: The Division Algorithm

The **Division Algorithm** says that given two integers a and b (with $b \neq 0$), there exist unique integers q and r such that $a = b \cdot q + r$ and $0 \leq r < b$. (You may think of q as the quotient and r as the remainder from your elementary school days.)

Also, we say that a set S is **closed under the Division Algorithm** if for every pair of integers a and b where $a \in S$ and $b \in S$, the integers q and r guaranteed by the Division Algorithm are also in S. If a set S is closed under the Division Algorithm, we call S a *DA-set*.

Suppose that we are given two sets S and T with $S \subseteq T$. Suppose also that for every $x \in T$, if $x \notin S$, then x is greater than every element of S. Then we say that $S \preccurlyeq T$.

P-1. Compute the values of q and r guaranteed by the Division Algorithm for the following pairs of integers:

a. $a = 32$ and $b = 4$. [1 pt]

b. $a = 27$ and $b = 5$ [2 pts]

c. $a = -325$ and $b = 7$ [2 pts]

P-2. Show that the sets $\{0, 1\}$ and $\{0, 1, 2, 3, 5\}$ are DA-sets. [5 pts]

P-3. Suppose that $S = \{0, 1, 2, 3, 4, 6\}$ and that T is a DA-set with $S \preccurlyeq T$. There are some integers that cannot be elements of T. For example, no integer in the interval $[15, 17]$ can be an element of T.

a. Explain why no integer in the interval $[15, 17]$ can be an element of T. [2 pts]

b. Find another interval $[m, n]$ where $n \geq m + 3$ such that no integer in $[m, n]$ can be an element of T. [3 pts]

P-4. Let $S = \{0, 1, 2, 3, 4, 5, 6, b\}$ with $b \geq 8$, and suppose that T is a DA-set with $S \preccurlyeq T$. There are some integers that cannot be elements of T. Show that no integer in the interval $[150, 154]$ can be an element of T. [5 pts]

P-5. Prove that if $S \preccurlyeq T$ and there are four elements of S and six elements of T, there is exactly one set U with five elements such that $S \preccurlyeq U \preccurlyeq T$. [5 pts]

P-6. Suppose that S is a DA-set where the elements are written in increasing order; that is, $S = \{0, 1, s_1, s_2, s_3, ..., s_n, s_{n+1}, ..., s_m\}$. Suppose n is a positive integer and that the DA-set S contains at least $n + 3$ elements, as described. Prove that $s_{n+1} \leq s_1 \cdot s_n + 1$. [5 pts]

P-7. Let $n \geq 1$, $a \geq 2$, and $S = \{0, 1, a, a^2, a^3, \cdots, a^n, b\}$ be a DA-set where the elements are written in increasing order (that is, $b > a^i$ for all of the a^i's in S). Suppose $b \neq 3$ and $b \neq a^{n+1}$. Prove that $b = a^n + 1$ or $b = a^{n+1} + 1$. [5 pts]

P-8. Prove that the set of whole numbers $\mathbb{W} = \{0, 1, 2, 3, \cdots\}$ is a DA-set. [5 pts]

P-9. a. Prove that if one whole number $w > 1$ is removed from the set of whole numbers $\mathbb{W} = \{0, 1, 2, 3, \cdots\}$, the resulting set $W' = W - \{w\}$ is not a DA-set. [3 pts]

b. Prove that if the positive multiples of some whole number $w > 1$ are removed from the set of whole numbers $\mathbb{W} = \{0, 1, 2, 3, \cdots\}$, the resulting set $W'' = W - \{kw\}$ is not a DA-set. [2 pts]

P-10. Suppose that the set $P = \{0, 1, 2, 4, 8, \cdots\}$ is a DA-set. Note that the numbers 3, 5, 6, and 7 are not elements of P (that is, the elements are written in increasing order). Find the other elements in the infinitely large set P. Justify that these are the other elements in P. [5 pts]

Solutions to 2016 Power Question

P-1. a. Because $32 = 8 \cdot 4 + 0$, $q = 8$ and $r = 0$.

b. Because $27 = 5 \cdot 5 + 2$, $q = 5$ and $r = 2$.

c. Because $-325 = -47 \cdot 7 + 4$, $q = -47$ and $r = 4$

P-2. The set $\{0, 1\}$ is a DA-set because $1 = 1 \cdot 1 + 0$ and $0 = 0 \cdot 1 + 0$, and both quotients and both remainders are in the set.

The set $\{0, 1, 2, 3, 5\}$ is a DA-set. There are 20 choices of (a, b) to consider. Notice that $5 = 1 \cdot 5 + 0$, $5 = 1 \cdot 3 + 2$, $5 = 2 \cdot 2 + 1$, $5 = 5 \cdot 1 + 0$, $3 = 0 \cdot 5 + 2$, $3 = 1 \cdot 3 + 0$, $3 = 1 \cdot 2 + 1$, $3 = 3 \cdot 1 + 0$, $2 = 0 \cdot 5 + 2$, $2 = 0 \cdot 3 + 2$, $2 = 1 \cdot 2 + 0$, $2 = 2 \cdot 1 + 0$, and all quotients and all remainders are in the set. Also, if $b = 1$ or $b = 0$, the quotients and remainders are in the set.

P-3. a. Because for all integers x in $[15, 17]$, $\lfloor x/3 \rfloor = 5$, so $q = 5$ will not be in T.

b. Answers will vary, but the explanations will be similar to the previous explanation. Examples of such intervals include $[20, 23]$, $[30, 35]$, and $[120, 143]$.

P-4. Notice that for $x \in [150, 154]$, $\lfloor \frac{x}{5} \rfloor = 30$. This implies that $30 \in T$. Notice also that $\lfloor \frac{30}{4} \rfloor = 7$, and $7 \notin T$. Therefore, no integer in $[150, 154]$ is in T.

P-5. Let T contain $\{s_1, s_2, s_3, s_4, x, y\}$ with $s_1 < s_2 < s_3 < s_4 < x < y$. The set S contains the s_i's. We will show that U must be the set $\{s_1, s_2, s_3, s_4, x\}$. First, $S \preccurlyeq U$ because $S \subseteq U$ and x is greater than each of the s_i's. Also, $U \preccurlyeq T$ because $U \subseteq T$ and y is greater than each of the s_i's and also x. If there were some other set of cardinality 5, it would have to be $U' = \{s_1, s_2, s_3, s_4, z\}$ for some z. The z must be chosen from the set $T - S$ and every element in $T - U'$ would have to be greater than z. This forces z to be equal to x, and the result is achieved.

P-6. First, we will show that $s_{n+1} < s_1(s_n + 1)$. Suppose otherwise; then the quotient $\lfloor \frac{s_{n+1}}{s_1} \rfloor$ is trapped strictly between s_n and s_{n+1}, but the set contains no elements between those two integers. If $x \in [s_1 \cdot s_n + 2, s_1(s_n + 1) - 1]$, then the remainder when dividing x by s_1 is an element of $[2, s_1 - 1]$, but that is an impossibility. Thus, the result is proven.

P-7. We cannot have $b = 3$ or $b = a^{n+1}$ by hypothesis. We consider the other possibilities.
If $b \in [a^n + 2, a^n + a - 1]$, then the remainder is in $[2, a - 1]$, but no such number exists in S.
If $b \in [a^n + a, a^{n+1} - 1]$, then the quotient $\lfloor \frac{b}{a} \rfloor \in [a^{n-1} + 1, a^n - 1]$, but no such number exists in S.
The case $b > a^{n+1} + 1$ is impossible from the result of P-6. Thus, the proof is complete.

P-8. Given two whole numbers a and b with $b \neq 0$, the quotient and remainder will both be whole numbers. Therefore, the quotient and remainder will be in the set $\mathbb{W}$. Thus, the set $\mathbb{W}$ is a DA-set.

P-9. a. Let w be the number removed from the set. Then, since $w+1$ and $w(w+1)$ are both in the set, $\dfrac{w(w+1)}{w+1} = w$ should be in the set. This is a contradiction, and thus it is impossible to remove just one whole number from the set of whole numbers and have a DA-set.

b. The number $2w+1$ is in W'', but $\lfloor \frac{2w+1}{w+1} \rfloor = 1$ and the remainder is w, and w is not in W''.

Note: This helps establish that if an infinite DA-set looks almost like the whole numbers, it must be the whole numbers.

P-10. The set P is the set containing all powers of 2 greater than 1 and also the number 0. If the integer p is of the form 2^k, then $\dfrac{2^k}{2^m}$ will be in P for all $m \leq k$ and the remainder is 0, which is also in P. If p is not of the form 2^k, then p satisfies $2^j < p < 2^{j+1}$ for some integer j. For either $2^{j+1} \div p$ or $p \div 2^j$, the remainder will be trapped between two powers of 2 and thus not in P.

Note: The only two types of infinite DA-sets are the set of whole numbers and sets like P in P-10. There are no other kinds of infinite DA-sets.

This question was inspired by the recently published paper Sets Closed Under the Division Algorithm by Robert O. Stanton, published in the American Mathematical Monthly (November 2015).

2016 Relay Problems

R1-1. The graph of $y = x^2$ has a minimum point A. The graph of $y = 128 - x^2$ has a maximum point Q. The graphs of $y = x^2$ and $y = 128 - x^2$ intersect at points U and D. Compute the area of $QUAD$.

R1-2. *Let N be the number you will receive.* Compute $\lfloor \log_3 N \rfloor$, which is the greatest integer less than or equal to $\log_3 N$.

R1-3. *Let N be the number you will receive.* The perimeter of $\triangle NYS$ is $3 \cdot N$. The lengths of all sides are positive integers. If $NY = YS$ and NS is as small as possible, compute the area of $\triangle NYS$.

R2-1. Suppose the ten digits are listed in the alphabetical order of their names in English. *For example, 8 comes before 5 because EIGHT appears in the dictionary before FIVE.* Which digit is fourth in the list?

R2-2. *Let N be the number you will receive.* The line $3x + y + N = 0$ intersects the line $4x + y = N$ at the point (a, b). Compute $a - b$.

R2-3. *Let N be the number you will receive.* Compute the number of times that the graph of $y = (x^2 - 9)^2 - N$ intersects the x-axis.

2016 Relay Answers

R1-1. 1024

R1-2. 6

R1-3. $3\sqrt{7}$

R2-1. 9

R2-2. 81

R2-3. 3

2016 Relay Solutions

R1-1. The point A is $(0,0)$. The point Q is $(0,128)$. The points U and D are $(\pm 8, 64)$. Notice that $QUAD$ is a rhombus, whose area is half the product of the diagonals: $\frac{1}{2} \cdot 16 \cdot 128 = \mathbf{1024}$.

R1-2. Because $\lfloor \log_3 N \rfloor = k$ if and only if $3^k \le N < 3^{k+1}$, compute powers of 3 while waiting. Notice that $3^6 = 729$ and $3^7 = 2187$, so the answer is $\mathbf{6}$.

R1-3. If $3N$ is odd, then $3N = 2k+1$ for some k, and $NY = YS = k$ and $NS = 1$, in which case the area of $\triangle NYS$ is $\frac{1}{2} \cdot 1 \cdot \sqrt{k^2 - (\frac{1}{2})^2} = \frac{1}{4}\sqrt{4k^2 - 1}$. If $3N$ is even, then $3N = 2k + 2$ for some k, and then $NY = YS = k$ and $NS = 2$ for an area of $\frac{1}{2} \cdot 2 \cdot \sqrt{k^2 - 1^2} = \sqrt{k^2 - 1}$. Because $3N = 18 = 2(8) + 2$, the area of $\triangle NYS$ is $\sqrt{8^2 - 1} = \sqrt{63} = \mathbf{3\sqrt{7}}$.

R2-1. The list of digits in their alphabetical order is 8, 5, 4, 9, 1, 7, 6, 3, 2, 0. The fourth digit in the list is $\mathbf{9}$.

R2-2. Because the lines cross at (a, b), it is true that $3a + b + N = 0$ and $4a + b - N = 0$. Adding these equations obtains the result that $7a+2b = 0 \to b = -\frac{7}{2}a$. Therefore, $4a - \frac{7}{2}a - N = 0 \to a = 2N$ and $b = -7N$. Thus, $a - b = 2N - (-7N) = 9N$. Substituting, $a - b = 9 \cdot 9 = \mathbf{81}$.

R2-3. The graph of $y = (x-3)^2(x+3)^2$ is a "big rounded W" touching the x-axis at $x = \pm 3$ with the center of the W topping off at $(0, 81)$. Because $y = (x^2 - 9)^2 - N$ touches the x-axis precisely when $(x^2 - 9)^2 = N$, there are:
zero intersection points if $N < 0$
two intersection points if $N = 0$
four intersection points if $0 < N < 81$
three intersection points if $N = 81$
two intersection points if $N > 81$
Because $N = 81$, there are $\mathbf{3}$ intersection points.

2016 Tiebreaker Problems

TB-1. If we define $i = \sqrt{-1}$, the summation $\sum_{n=1}^{n=32} (1+i)^n = k(1-i)$. Compute k.

TB-2. There are N right triangles for which all of the side lengths are integers and for which one of the legs has length 60. Compute N.

2016 Tiebreaker Answers

TB-1. 65535

TB-2. 13

2016 Tiebreaker Solutions

TB-1. Powers of i cycle in blocks of 4, so consider $(1+i)^0 + (1+i)^1 + (1+i)^2 + (1+i)^3 = 1+1+1+2i+2i(1+i) = 5i$.

Expressions of the form $(1+i)^k + (1+i)^{k+1} + (1+i)^{k+2} + (1+i)^{k+3}$ simplify to $(1+i)^k \cdot 5i$. For $k = 1, 5, 9, ..., 29$, there are 8 blocks of 4 terms which when added are equivalent to the given summation.

With a little effort, $k = 1, 5, 9, 13$ produce $-5(1-i)$, $20(1-i)$, $-80(1-i)$, and $320(1-i)$, so we see that the sequence is geometric with common ratio -4. Thus the required sum is $\dfrac{a(1-r^n)}{1-r} = \dfrac{-5(1-(-4)^8)}{1-(-4)} = 4^8 - 1 = \mathbf{65535}$.

TB-2. Let the other sides be x and z with $x < z$. Then, $x^2 + 60^2 = z^2 \rightarrow 3600 = (z-x)(z+x)$ where z and x are of the same parity. Let $z - x = 2a$ and $z + x = 2b$. Then the problem situation yields $900 = ab$ where $a < b$. Since $900 = 2^2 \cdot 3^2 \cdot 5^2$, the number of divisors is $(2+1)(2+1)(2+1) = 27$, and there are 13 possible values of a (the smaller of the two divisors). Therefore, there are **13** triangles.

2017 Contest at Stuyvesant High School (New York City)

Contents

2017 Team Problems

T-1. Compute the number of positive integers that are less than 2017 and that are not multiples of 20 or 17.

T-2. A group of 2017 students is having a single-elimination Rubik's cube solving tournament. In any round where an odd number of students begins the round (like the first round), one student is given a bye. How many rounds will have a bye in this 2017-person tournament? *When a player is given a bye, that player does not play against an opponent in that particular round. The player who is given the bye automatically advances to the next round.*

T-3. When written as a decimal, $\dfrac{20}{17} = 1.176\ldots$. Compute the sum of the first 2017 digits of the decimal expansion of $\dfrac{20}{17}$.

T-4. A 3-by-3-by-3 wooden cube is painted red on the outside and is then cut into 27 unit cubes. A unit cube is selected uniformly at random and placed, at random, on a table. Suppose the top face of that unit cube is red. Let P_1 be the probability that exactly one other face of that unit cube is red. Let P_2 be the probability that exactly two other faces of that unit cube are red. Compute $P_1 - P_2$.

T-5. The letters of the word MATHEMATICA are written on eleven tiles, one per tile. Four of those eleven tiles are chosen uniformly at random without replacement. Compute the probability that there are two distinct vowels and two consonants written on the four tiles chosen.

T-6. Nonconvex hexadecagon $P_1P_2P_3\cdots P_{16}$ has the following properties:

- Interior angles have measure $\mathrm{m}\angle P_n = 45^\circ$ if n is odd and $\mathrm{m}\angle P_n = 270^\circ$ if n is even.
- Sides which have P_4, P_8, P_{12}, or P_{16} as an endpoint have length 1.
- Sides which have P_2, P_6, P_{10}, or P_{14} as an endpoint have length $\sqrt{2}$.

Compute the area of the hexadecagon.

T-7. Let f, g, and h be distinct linear functions of x for which $f(6) = g(6) = h(6) = 6$ and whose roots are the positive integers a, b, and c respectively. The product of the slopes of the graphs of the three functions equals -1. Compute the maximum possible value of the product abc.

T-8. Compute the number of lattice points (i.e., points with integer coordinates) which are on the graph of $x^2 + xy - 3x + 2y - 2016 = 0$ and which lie in the first quadrant.

T-9. For two ordered triples of distinct positive integers (x, y, z) with $1 < x < y < z$, it is true that $x + y + z = 180$, x and y have the same parity, x and z have opposite parity, y is a multiple of x, and z is a multiple of y. Compute both ordered triples.

T-10. A fair coin is flipped nine times in succession. Define a *run* to be a maximal sequence of <u>three</u> or more consecutive matching flips, like TTT or HHHH. Compute the probability that the sequence of flips does **not** contain a run.

2017 Team Answers

T-1. 1803

T-2. 5

T-3. 9073

T-4. 0

T-5. $\dfrac{7}{11}$

T-6. 8

T-7. 6384

T-8. 2

T-9. $(5, 25, 150)$ and $(5, 35, 140)$ *[must have both]*

T-10. $\dfrac{55}{256}$

2017 Team Solutions

T-1. Use the Principle of Inclusion-Exclusion. There are 2016 positive integers that are less than 2017. Of those, $\lfloor \frac{2017}{20} \rfloor = 100$ are multiples of 20 and $\lfloor \frac{2017}{17} \rfloor = 118$ are multiples of 17, leaving $2016 - 100 - 118 = 1798$. However, the multiples of $20 \cdot 17$ were removed twice, so add them back in. There are $\lfloor \frac{2017}{340} \rfloor = 5$ of those, so the answer is $2016 - 100 - 118 + 5 = \mathbf{1803}$.

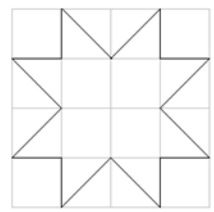
T-2. The first round will have a bye, and $2018 \div 2 = 1009$ students will remain after the first round. There will be a bye in the second round, and $1010 \div 2 = 505$ students will remain after the second round. A bye will be awarded in the third round, and $506 \div 2 = 253$ students will remain after the third round. A bye will be given in the fourth round, and $254 \div 2 = 127$ students will remain after the fourth round. A bye will be awarded in the fifth round, and $128 \div 2 = 64$ students will remain after the fourth round. No bye will be awarded thereafter, because 64, 32, 16, 8, 4, and 2 are powers of 2. Thus, there will be **5** byes.

T-3. The repetend is 1764705882352941, which has sum 72 and 16 digits. Because 2017 can be expressed as $2017 = 126 \cdot 16 + 1$, the repetend appears 126 times after the initial 1, and that makes 2017 digits. The desired sum is $126 \cdot 72 + 1 = \mathbf{9073}$.

T-4. The 27 cubes have a total of $6 \times 3 \times 3 = 54$ red faces. One of these is the top of the chosen cube. Twelve of the cubes have two faces painted red, so in $12 \times 2 = 24$ of the 54 cases, exactly one of the other five faces is red. Eight of the cubes have three faces painted red, so in $8 \times 3 = 24$ of the 54 cases, exactly two of the other five faces are red. Therefore, the probabilities of the two described events are equal. The difference is **0**.

T-5. First, a 4-letter selection with no restrictions can be made in $\binom{11}{4} = \dfrac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 10 \cdot 3$ ways. A 4-letter selection with two distinct vowels and two consonants must contain A, E or A, I, or E, I with two consonants. Choosing A, E, and two consonants or A, I, and two consonants can be done in $3 \cdot 1 \cdot 6 \cdot 5$ ways. Choosing E, I, and two consonants can be done in $1 \cdot 1 \cdot 6 \cdot 5$ ways. The desired probability, therefore, is $\dfrac{2 \cdot 3 \cdot 1 \cdot 6 \cdot 5 + 1 \cdot 1 \cdot 6 \cdot 5}{11 \cdot 10 \cdot 3} = \dfrac{5 \cdot 6 \cdot 7}{11 \cdot 10 \cdot 3}$, or $\mathbf{\dfrac{7}{11}}$.

T-6. The hexadecagon is drawn below. It is a union of many 45-45-90 triangles, as shown.



Superimposing it on a grid of unit squares, it can be determined that its area is $4+8\cdot\frac{1}{2}=\mathbf{8}$.

T-7. Because the product of the slopes of the graphs of the three functions is negative, either one of a, b, and c is negative or all three are negative. Because of symmetry about $x=6$, the product will be maximized if all three slopes are negative. The slope of any such linear equation with a root r greater than $x=6$ is $m=\frac{-6}{r-6}$. Using roots $a=7$ and $b=8$ yields slopes of $m_f=-6$ and $m_g=-3$ respectively. The slope $m_h=\frac{-1}{18}\rightarrow c=114$. The maximum product abc is $7\cdot 8\cdot 114=\mathbf{6384}$.

Note: we should at least consider how not using the two smallest positive denominators for $m=\frac{-6}{r-6}$ affects the product abc. Consider trying to get viable integer roots as close to each other as possible. Because $m_f\cdot m_g\cdot m_h=\left(\frac{-6}{a-6}\right)\cdot\left(\frac{-6}{b-6}\right)\cdot\left(\frac{-6}{c-6}\right)=-1$, we know that $(a-6)(b-6)(c-6)=216$. Solutions such as $(a,b,c)=(15,12,10)$ yield products that are much less than 6384. Even the next closest case to the optimal case is $(a,b,c)=(7,9,78)$, and that product is $abc=4914$, which is substantially less than 6384.

T-8. Collect terms with y on one side and terms without y on the other: $y(x+2)=-x^2+3x+2016$. This implies that $y=\frac{-x^2-2x}{x+2}+\frac{5x+10}{x+2}+\frac{2006}{x+2}=-x+5+\frac{2006}{x+2}$. For both x and y to be integers, $x+2$ must be a factor of 2006. The factors of 2006 are ±1, ±2, ±17, ±34, ±59, ±118, ±1003, and ±2006. Reject the negative values because x must be positive. Also, $x+2$ cannot be 1 or 2 for the same reason. Thus x must come from the set $\{15,32,57,116,1001,2004\}$. But y must also be positive, which eliminates $x=2004$, $x=1001$, $x=116$, and $x=57$. The remaining values of x give positive values of x and y, and are $x=15$ and $x=32$. There are **2** such values.

This is a "NYSML Classic". It is very much like question T9 from NYSML1997. Good math problems stand the test of time!

T-9. To obtain a sum of 180, x and y must be odd and z must be even. Because $1 < x < y < z$, $x \geq 3$, and $y \geq x + 2$. Let $y = ax$ for some integer a. Because x and y are odd, a must also be odd. Similarly, let $z = by$ for some integer b. Because y is odd and z is even, b must be even. Because $x + y + z = 180$, $x + ax + abx = 180$, or $x = \dfrac{180}{1 + a + ab}$. The denominator $1 + a + ab$ must be an even factor of 180 and produce an odd x-value. The only even factors of 180 that produce odd x-values are 4, 12, 20, 36, 60, and 180, but 4 and 180 are easily eliminated. Consider the following:
If $a + ab = 11$, then $a(1 + b) = 11$, which has no solutions with $a > 1$ and $b \neq 0$.
If $a + ab = 19$, this case is similar.
If $a + ab = 35$, then $a(1 + b) = 35$, which implies $a = 5$ and $b = 6$ or $a = 7$ and $b = 4$. These generate the ordered triples $(5, 25, 150)$ and $(5, 35, 140)$.
If $a + ab = 59$, then this case is similar to the first two cases.
The answers are $(\mathbf{5}, \mathbf{25}, \mathbf{150})$ and $(\mathbf{5}, \mathbf{35}, \mathbf{140})$.

T-10. There are $2^9 = 512$ possible sequences, so count the number of run-free sequences by means of recurrence relations. Let $S(n)$ represent the number of run-free sequences of length n with the last two flips the same and $D(n)$ represent the number of run-free sequences of length n with the last two flips different. By inspection, $S(2) = 2$ [these are HH and TT] and $D(2) = 2$ [these are HT and TH].
Now, consider $S(n + 1)$ and $D(n + 1)$. This problem concerns itself with run-free sequences, so assume that the first n flips in the $n+1$ are run-free, and check the final three flips of each sequence.
Consider $S(n + 1)$. If the first n flips end in two that match, it's impossible to flip the coin one more time and end with the last two flips the same, since that would produce a run of 3 at the end. If the first n flips end with the last two different, then the last two flips are the same if the $n + 1^{\text{st}}$ flip matches the n^{th} flip. Therefore, $S(n + 1) = D(n)$.
Consider $D(n+1)$. Starting with a run-free sequence of n flips, mismatching the final flip will always produce a sequence that ends in two different flips. This means that for any run-free length-n sequence, there is one length-$(n + 1)$ sequence ending with two different flips, and thus $D(n + 1) = S(n) + D(n)$.
Using these relations, construct the following table.

n	2	3	4	5	6	7	8	9
$S(n)$	2	2	4	6	10	16	26	42
$D(n)$	2	4	6	10	16	26	42	68
$S(n) + D(n)$	4	6	10	16	26	42	68	110

So the probability of a length-9 run-free sequence is $\dfrac{S(9) + D(9)}{2^9} = \dfrac{110}{512} = \mathbf{\dfrac{55}{256}}$.

Author's Note: The probability is based on $S(n) + D(n)$, which we found to be $D(n+1)$. However, we also know that $S(n) = D(n-1)$, so $D(n+1) = D(n) + D(n-1)$, meaning that the $D(n)$ row is a Lucas (Fibonacci-like) sequence starting with $D(2) = 2$ and $D(3) = 4$. Backtracking produces $D(1) = 2$, confirming the fact that $D(n)$ is twice the n^{th} Fibonacci number. We can work this out another way. A run-free sequence of length n will consist of an alternating series of runs of H and T each of length 1 or 2. This is isomorphic to the famous "staircase problem" which counts the number of ways to climb a staircase of n steps going up either 1 or 2 steps at a time. The solution (omitted here) is that the number of ways for n steps is the $n + 1^{\text{st}}$ Fibonacci number. Since any run-free sequence can start with either an H or a T, the number of such sequences is double a Fibonacci number, so $P(n) = \dfrac{2F_{n+1}}{2^n} = \dfrac{F_{n+1}}{2^{n-1}}$.

2017 Individual Problems

I-1. In a sequence of seven consecutive integers, the sum of the third and fourth integers is 2017. Compute the sum of the seven consecutive integers.

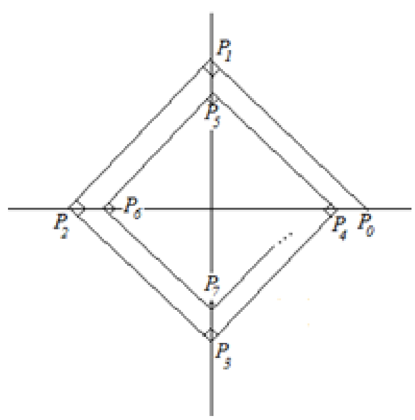
I-2. Compute the ordered triple of positive integers (x, y, z) that satisfies $2^x + 2^y = 2^z$ and $x + y + z = 2017$.

I-3. Compute the number of ordered pairs of relatively prime positive integers (a, b) where $a < b$ such that $a + b = 2017$.

I-4. In hexagon $HEXAGN$, all sides have length 4, m$\angle H = m\angle X =$ m$\angle G = 90°$, and $\angle E \cong \angle A \cong \angle N$. Compute the area of $HEXAGN$.

I-5. In $\triangle ABC$, medians $\overline{AD}$ and $\overline{BE}$ are perpendicular to each other. Given that $AC = 6$ and $BC = 8$, compute AB.

I-6. A path $P_0P_1P_2\ldots$ is constructed in the xy-plane as follows. Begin at $P_0(1, 0)$ and draw a line segment with slope m where $-1 < m < 0$ to the point P_1 on the y-axis. Then, construct a segment perpendicular to $\overline{P_0P_1}$ through P_1, intersecting the negative x-axis at P_2.



Repeat this process indefinitely, continuing straight between P_i's and making 90° left turns at the coordinate axes, as shown above. Given that the total length of the path is 29, compute m.

I-7. Compute the sum of all integer values of k for which $8^{(k+6)/(2k-3)}$ is an integer less than 10.

I-8. For some real number a, let r_1, r_2, and r_3 be the not-necessarily-distinct roots of $x^3 + 6x^2 - ax + 3 = 0$. Compute the minimum possible value of

$$\left(r_1 + \frac{1}{r_1}\right)^2 + \left(r_2 + \frac{1}{r_2}\right)^2 + \left(r_3 + \frac{1}{r_3}\right)^2.$$

I-9. Define the function f on the nonnegative integers by $f(n) = n$ for $0 \leq n \leq 20$, and $f(n)$ is the greater of $(f(n-20)+17)$ and $(f(n-17)+20)$ if $n > 20$. Compute $f(2017)$.

I-10. Toni is exactly 3 miles from her destination, driving at 55 miles per hour. If she were instead to drive the 3 miles at k miles per hour faster, she would arrive one minute sooner. Compute k.

2017 Individual Answers

I-1. 7063

I-2. $(672, 672, 673)$

I-3. 1008

I-4. $24 + 8\sqrt{3}$

I-5. $2\sqrt{5}$

I-6. $-\dfrac{20}{21}$

I-7. 6

I-8. 29

I-9. 2371

I-10. $\dfrac{121}{5}$ or $24\dfrac{1}{5}$ or 24.2

2017 Individual Solutions

I-1. Let the seven integers be $x-3$, $x-2$, $x-1$, x, $x+1$, $x+2$, and $x+3$. Then, according to the problem statement, $x+x-1=2017$, so $2x-1=2017 \rightarrow x=1009$. The sum of the seven consecutive integers is $7x=7(1009)=\mathbf{7063}$.

I-2. Factor to obtain $2^x(1+2^{y-x})=2^z$. Since the right side is a power of 2, the expression $1+2^{y-x}$ must be a power of 2, and that can only happen if $2^{y-x}=1 \rightarrow y-x=0$. This gives $2^x(1+1)=2^{x+1}=2^z$, so $z=x+1$. Now, solve $x+x+x+1=2017$ to obtain $x=672$. The ordered triple is $(\mathbf{672},\mathbf{672},\mathbf{673})$.

I-3. If a and b are not relatively prime, then they have some common factor g, where $g>1$. If $g \mid a$ and $g \mid b$, then $g \mid (a+b) \rightarrow g \mid 2017$. But 2017 is prime, so the only possible value for g is 2017, in which case $a+b$ cannot equal 2017. Thus, for every ordered pair of positive integers (a,b) with $a,b<2017$, a and b will be relatively prime. Because $a<b$, a can take on every integer value from 1 to 1008, so there are **1008** different ordered pairs.

I-4. The measures of angles E, A, and N are each one-third of $720^\circ - 3\cdot 90^\circ = 450^\circ$, or 150°. Draw in diagonals $\overline{EA}$, $\overline{AN}$, and $\overline{NE}$. Notice that $\triangle EAN$ is equilateral with side length $\sqrt{4^2+4^2}=4\sqrt{2}$. Thus, $[HEXAGN]$ is equal to $3\left(\frac{1}{2}\cdot 4\cdot 4\right)+\frac{(4\sqrt{2})^2\cdot\sqrt{3}}{4}$, or $\mathbf{24}+\mathbf{8}\sqrt{\mathbf{3}}$.

This is a "NYSML Classic". It is very much like question I6 from NYSML2002. Shapes are fun in any era!

I-5. Let F be the point of intersection of the medians and let $DF=x$ and $EF=y$. Because $\overline{AD}$ and $\overline{BE}$ are medians, $AF=2x$ and $BF=2y$. Applying the Pythagorean Theorem on $\triangle AFE$ and $\triangle BFD$ yields $4x^2+y^2=9$ and $x^2+4y^2=16$ respectively. Adding the two equations yields $5x^2+5y^2=25$. Applying the Pythagorean Theorem to $\triangle AFB$, $AB^2=4x^2+4y^2=\frac{4}{5}\cdot 25=20$, so $AB=\sqrt{20}=\mathbf{2}\sqrt{\mathbf{5}}$.

I-6. The slope of $\overline{P_0P_1}$ is m, so if P_1 has coordinates $(0,y)$, $\frac{y-0}{0-1}=m \rightarrow y=-m$. Now, because $P_1(0,-m)$ and the slope of $\overline{P_1P_2}$ is $-\frac{1}{m}$, the coordinates of P_2 can be found in a similar way to be $(-m^2,0)$. Let O be the origin. Notice that $\triangle OP_0P_1 \sim \triangle OP_1P_2 \sim \triangle OP_2P_3 \sim \cdots$. That implies that the lengths of the hypotenuses of the right triangles form a geometric series, and because the lengths of OP_0, OP_1, and OP_2 are 1, $-m$, and m^2, the common ratio of the geometric series is $-m$.
For an infinite geometric series with first term a_1 and common ratio r, the sum is $S=\frac{a_1}{1-r}$.

By the Pythagorean Theorem, $P_0P_1 = \sqrt{m^2+1}$, so solve $29 = \dfrac{\sqrt{m^2+1}}{1-(-m)} \to m^2+1 = 29^2(m^2+2m+1)$, which is equivalent to $840m^2+2\cdot 841m+840 = 0 \to 420m^2+841m+420 = 0$. This factors as $(20m+21)(21m+20) = 0$, which has only one value of m in the desired interval, $m = -\dfrac{\mathbf{20}}{\mathbf{21}}$.

I-7. The possible exponents are 0, 1, 1/3, and 2/3 because if 8^a equals some number other than 1, 2, 4, or 8, the value of k is transcendental. If $\dfrac{k+6}{2k-3} = 0$, then $k = -6$. If $\dfrac{k+6}{2k-3} = 1$, then $k = 9$. If $\dfrac{k+6}{2k-3} = \dfrac{1}{3}$, then $k = -21$. If $\dfrac{k+6}{2k-3} = \dfrac{2}{3}$, then $k = 24$. The sum of these values is $-6+9-21+24 = \mathbf{6}$.

I-8. The expression is symmetric, so we use symmetric elementary functions. Notice that $\left(r_1+\dfrac{1}{r_1}\right)^2 + \left(r_2+\dfrac{1}{r_2}\right)^2 + \left(r_3+\dfrac{1}{r_3}\right)^2 = (r_1^2+r_2^2+r_3^2) + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} + 6 = (r_1+r_2+r_3)^2 - 2(r_1r_2+r_1r_3+r_2r_3) + \left(\dfrac{(r_1r_2)^2+(r_1r_3)^2+(r_2r_3)^2}{(r_1r_2r_3)^2}\right) + 6$. By Viete's formulas, this is equal to $(-6)^2 - 2(-a) + \left(\dfrac{(r_1r_2+r_1r_3+r_2r_3)^2 - 2(r_1r_2r_3)(r_1+r_2+r_3)}{(-3)^2}\right) + 6 = 42 + 2a + \left(\dfrac{(-a)^2 - 2(-3)(-6)}{9}\right) = 42 + 2a + \dfrac{a^2}{9} - 4 = \dfrac{1}{9}(a^2+18a+342) = \dfrac{1}{9}((a+9)^2+261) = \dfrac{1}{9}(a+9)^2 + 29$. Because the quadratic term is minimized when $a = -9$, the minimal value is $\mathbf{29}$.

I-9. Find values for $f(21)$, $f(22)$, and so on, and note that $f(n) = n+3k$ for values of k that satisfy $20+(k-1)(17) \le n \le 20+17k$. Notice that $2026 = 20+118(17)$, so $f(2026) = 2380$, and thus $f(2017) = f(2026-9) = 2380-9 = \mathbf{2371}$.

I-10. Let r and d denote the original speed and distance to the destination, respectively. At this rate, Toni reaches her destination in $t = d/r$ hours. If she increases her rate to $r+k$ miles per hour, her new time is $t = \dfrac{d}{r} - \dfrac{1}{60}$ hours, or $\dfrac{60d-r}{60r}$ hours. Equating and substituting, $r+k = \dfrac{d}{\frac{60d-r}{60r}} = \dfrac{60rd}{60d-r}$. Now, substituting, $r+k = \dfrac{60\cdot 55\cdot 3}{60\cdot 3-55} = \dfrac{60\cdot 11\cdot 3}{25} = \dfrac{396}{5} = 79.2$ miles per hour. Thus, $k = 79.2-55 = \mathbf{24.2}$. *Author's note: Slow down and enjoy the scenery!*

Power Question 2017: Quaternions

The *quaternions* are numbers of the form $a + bi + cj + dk$, where a, b, c, d are real and the constants i, j, k have the property that $i^2 = j^2 = k^2 = ijk = -1$. They are an extension of the complex numbers. The set is usually denoted $\mathbb{H}$ for William Rowan Hamilton who developed them in 1843. The addition of two quaternions u and v, where $u = u_1 + u_2 i + u_3 j + u_4 k$ and $v = v_1 + v_2 i + v_3 j + v_4 k$ is defined $u + v = u_1 + v_1 + (u_2 + v_2)i + (u_3 + v_3)j + (u_4 + v_4)k$. We also define multiplication by a real number c such that $cu = cu_1 + cu_2 i + cu_3 j + cu_4 k$ and $uc = u_1 c + u_2 ci + u_3 cj + u_4 ck$. Because of the commutative property of multiplication of real numbers, for the real number c and quaternion u, $cu = uc$. Defining multiplication of two quaternions will require some care because, although we will assume that multiplication of quaternions is associative, we do not assume that multiplication of quaternions is commutative.

P-1. a. Compute the sum $(3 - 2i - 4j + 6k) + (7 - 5i + 3j + 4k)$. [2 pts]

b. Find a quaternion z such that $z + (2 + 0i + 1j - 7k) = 0$. [3 pts]

P-2. a. If we right-multiply both sides of the equation $ijk = -1$ by k, we get $ijk^2 = -1k = -k$, or $ij(-1) = (-1)k$, so $ij = k$. Left-multiply both sides of the equation the equation $ijk = -1$ by i to show that $jk = i$. [3 pts]

b. Show that $kji = 1$. [3 pts]

P-3. Use the result of **P2** to show that $kj = -i \neq jk$, which establishes that multiplication of quaternions is not commutative. [4 pts]

Using similar procedures, it turns out that the following multiplication table is established.

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Multiplication can then be defined using the distributive property and the table above. If $u = u_1 + u_2 i + u_3 j + u_4 k$ and $v = v_1 + v_2 i + v_3 j + v_4 k$, by straightforward calculations one may establish the following identity:

$$\begin{aligned} u \times v = & u_1 v_1 - u_2 v_2 - u_3 v_3 - u_4 v_4 \\ & + (u_2 v_1 + u_1 v_2 - u_4 v_3 + u_3 v_4)i \\ & + (u_3 v_1 + u_4 v_2 + u_1 v_3 - u_2 v_4)j \\ & + (u_4 v_1 - u_3 v_2 + u_2 v_3 + u_1 v_4)k. \end{aligned}$$

P-4. a. Find the product $(3 + 2i - j)(4 - 2k)$. [2 pts]

b. Find the product $(4 - 2k)(3 + 2i - j)$. [2 pts]

P-5. Many of the properties of a field (your Abstract Algebra professor will explain this more some day) apply to $\mathbb{H}$. It is easy to show that the set $\mathbb{H}$ is closed under addition and multiplication (that is, the sum or product of two quaternions is a quaternion). Show that the following properties hold.

a. The set $\mathbb{H}$ has an additive identity and a multiplicative identity. *That is, there are quaternions a and m such that for all quaternions u, $a + u = u + a = u$ and $m \cdot u = u \cdot m = u$.* [2 pts]

b. The set $\mathbb{H}$ is commutative under addition. [2 pts]

c. The element v is such that $v = -u$. This can be proven by adding component-wise: $u + (-u) = (u_1 + u_2 i + u_3 j + u_4 k) + (-u_1 + -u_2 i + -u_3 j + -u_4 k)$, so $u + (-u) = 0 + 0i + 0j + 0k = 0$, and 0 is the additive identity. [2 pts]

P-6. If $u = u_1 + u_2 i + u_3 j + u_4 k$, define the *conjugate* $\overline{u}$ as $\overline{u} = u_1 - u_2 i - u_3 j - u_4 k$. Show that $u\overline{u} = \overline{u}u = u_1^2 + u_2^2 + u_3^2 + u_4^2$. Note that this is a nonnegative real number, and that it is the square of the distance from u to the origin when u is taken to be a point in four-dimensional space. The square root of this quantity, $\sqrt{u\overline{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$, is denoted $\|u\|$ and is called the *norm* of u. [4 pts]

P-7. For a nonzero quaternion $u = u_1 + u_2 i + u_3 j + u_4 k$, the *inverse* of u, written u^{-1}, is defined as the quaternion such that $uu^{-1} = u^{-1}u = m$, where m is the same value from question **P5-a**. Find an expression for u^{-1} in terms of the u_n and $\overline{u}$. [5 pts]

P-8. Show that, for nonzero quaternions u and v, the inverse of the product uv is given by $(uv)^{-1} = v^{-1}u^{-1}$. [5 pts]

P-9. Given a quaternion $u = u_1 + u_2 i + u_3 j + u_4 k$, we call the quaternion *pure* if $u_1 = 0$, and we call the quaternion a *unit quaternion* if $\|u\| = 1$.

a. Verify that all pure unit quaternions are square roots of -1. *Note that this implies that there are infinitely many square roots of -1.* [2 pts]

b. Prove that the only square roots of -1 are pure unit quaternions. [3 pts]

P-10. Determine, with proof or counterexample, whether the Zero Product Property is true for quaternions. That is to say, determine whether it is true that, given quaternions u and v such that $u \cdot v = 0$, then $u = 0$ or $v = 0$. [6 pts]

Solutions to 2017 Power Question

P-1. a. Adding components, this sum is $10 - 7i - j + 10k$.

b. This quaternion can be found by negating each coefficient, so $z = -2 - 1j + 7k$.

P-2. a. Left-multiplying obtains $i^2jk = i(-1) = -i$, which implies $(-1)jk = (-1)i$, and dividing by -1 obtains the result $jk = i$.

b. From **P2-a**, we have $ij = k$. Right-multiplying by j yields $ij^2 = kj \to -i = kj$. Now, right-multiplying by i yields $-i^2 = kji$, so $kji = (-1)(-1) = 1$, as needed.

P-3. Right-multiplying both sides of $kji = 1$ by i yields $kji^2 = i \to kj(-1) = i \to kj = -i$. Multiplication of quaternions is not commutative.

P-4. a. Using the distributive property and the table, $(3 + 2i - j)(4 - 2k) = 3 \cdot 4 + 2i \cdot 4 - j \cdot 4 + 3 \cdot -2k + 2i \cdot -2k + -j \cdot -2k$, or $12 + 8i - 4j - 6k + 4j + 2i$. Simplifying, this is $12 + 10i - 6k$.

b. Using the distributive property and the table, $(4 - 2k)(3 + 2i - j) = 4 \cdot 3 + 4 \cdot 2i + 4 \cdot -j + -2k \cdot 3 + -2k \cdot 2i + -2k \cdot -j$. Simplifying, this is $12 + 4i - 8j - 6k$.

P-5. a. The additive identity is 0 and the multiplicative identity is 1. The proofs are trivial.

b. Because addition of quaternions is done component-wise, and the addition of the real number coefficients is commutative, the addition of quaternions is commutative.

c. The element v is such that $v = -u$. This can be proven by adding component-wise: $u + (-u) = (u_1 + u_2i + u_3j + u_4k) + (-u_1 + -u_2i + -u_3j + -u_4k)$, so $u + (-u) = 0 + 0i + 0j + 0k = 0$, and 0 is the additive identity.

P-6. Applying the definition of multiplication and the definition of conjugate, $u\overline{u}$ is equal to $u_1^2 + u_2^2 + u_3^2 + u_4^2 + (u_1 \cdot -u_2 + u_2 \cdot u_1 + u_3 \cdot -u_4 - u_4 \cdot -u_3)i + (u_1 \cdot -u_3 - u_2 \cdot -u_4 + u_3 \cdot u_1 + u_4 \cdot -u_2)j + (u_1 \cdot -u_4 + u_2 \cdot -u_3 - u_3 \cdot -u_2 + u_4 \cdot u_1)k$. The coefficients of i, j, and k are all 0, yielding the desired result.

P-7. From **P6**, $u\overline{u} = u_1^2 + u_2^2 + u_3^2 + u_4^2$. Dividing both sides by $u_1^2 + u_2^2 + u_3^2 + u_4^2$ yields $u \cdot \dfrac{\overline{u}}{u_1^2 + u_2^2 + u_3^2 + u_4^2} = 1$. Therefore, $u^{-1} = \dfrac{\overline{u}}{u_1^2 + u_2^2 + u_3^2 + u_4^2}$. Note that $u^{-1} \cdot u = \dfrac{\overline{u}}{u_1^2 + u_2^2 + u_3^2 + u_4^2} \cdot u = \dfrac{u_1^2 + u_2^2 + u_3^2 + u_4^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2} = 1$, as required.

P-8. By definition of inverse, $(uv)(uv)^{-1} = 1$. Also, notice that $(uv)(v^{-1}u^{-1}) = u \cdot 1 \cdot u^{-1} = uu^{-1} = 1$. Because inverses of quaternions are unique, $(uv)^{-1} = v^{-1}u^{-1}$.

P-9. a. Let the pure unit quaternion be $u = 0 + u_2i + u_3j + u_4k$ where $u_2^2 + u_3^2 + u_4^2 = 1$. Applying the definition of multiplication of quaternions, $u^2 = 0^2 - u_2^2 - u_3^2 - u_4^2 + (0 + 0 + u_3u_4 - u_4u_3)i + (0 - u_2u_4 + 0 + u_4u_2)j + (0 + u_2u_3 - u_3u_2 + 0)k$, which implies $u^2 = -(u_2^2 + u_3^2 + u_4^2) = -1$. Thus, every pure unit quaternion is a square root of -1.

b. First, consider that if q is a nonzero pure quaternion, then q^2 is a negative real number. This can be proven by recognizing that (by **P6**) $q \cdot \overline{q}$ is a positive real number, and then by recognizing that if q is pure, then $\overline{q} = -q$, so $q \cdot -q$ is a positive real number, which implies that q^2 must be a negative real number.

Now, let $u^2 = -1$, and write $u = r + q$ where r is real and q is pure. Then, $u^2 = r^2 + 2rq + q^2$, or $u^2 = (r^2 + q^2) + 2rq$. Because q^2 is a negative real number, $r^2 + q^2$ is a real number. Also, $2rq$ is pure. But because $u^2 = -1$, $2rq = 0$, which implies $r = 0$ or $q = 0$. (Notice that this must be true because multiplication of a quaternion by a number is established in the Background, and so we don't need the Zero Product Property for quaternions yet.) If $q = 0$, then $u^2 = r^2 \geq 0$. This is a contradiction because $u^2 = -1$. Therefore, $r = 0$ and $u = q$. So, $q^2 = -q \cdot \overline{q} = -\|q\|^2 = -1$, and the norm of q equals 1, which implies that u is a pure unit quaternion.

P-10. The Zero Product Property does hold for quaternions. Consider that, from **P8**, the product of two nonzero quaternions has an inverse. If the product of two nonzero quaternions has an inverse, that product cannot be zero. Therefore, if the product of two quaternions is zero, one of them has to be zero.

2017 Relay Problems

R1-1. There are some people walking some dogs in a park. In all, the total number of people and dogs is 37. Each person has two legs and each dog has four legs. If the total number of legs on all of the creatures in the park is 108, compute the number of people in the park.

R1-2. *Let N be the number you will receive.* A triangle has perimeter N, and each of its sides has integral length. If the area of the triangle is as large as it can be (given these constraints), its area is $a\sqrt{b}$, where a and b are integers and b has no factor greater than 1 which is a perfect square. Pass back the ordered pair (a, b).

R1-3. *Let (a, b) be the ordered pair you will receive.* A circle of diameter a is tangent to sides $\overline{AB}$ and $\overline{AD}$ of square $ABCD$ with side length b. The diagonal $\overline{AC}$ of the square is broken into three segments by the circle. Two of these segments are outside the circle. Compute the length of the larger of these two segments.

R2-1. Triangle OPQ is given, with O at the origin, P on the x-axis, and Q on the y-axis. If the medians of $\triangle OPQ$ meet at $(5, 2)$, compute the area of $\triangle OPQ$.

R2-2. *Let N be the number you will receive.* If the reciprocal of the sum of the reciprocals of x and $\dfrac{N}{5}$ equals 12, compute x.

R2-3. *Let N be the integer you will receive.* Compute the greatest integer value of x that makes the expression $(x - N)^2 + 16$ a perfect square.

2017 Relay Answers

R1-1. 20

R1-2. $(6, 10)$

R1-3. $7\sqrt{2} - 3$

R2-1. 45

R2-2. -36

R2-3. -33

2017 Relay Solutions

R1-1. Solve the system $P+D=37$ and $2P+4D=108$ to obtain $P=20$ and $D=17$. The number of people is $\mathbf{20}$.

R1-2. Given a triangle of fixed perimeter, the largest area will come from the triangle that is closest to equilateral. That means there will be sides of 7, 7, and 6. This is isosceles, so drop an altitude to the side of length 6 and compute the height $=\sqrt{7^2-3^2}=2\sqrt{10}$. So the area of the triangle is $\frac{1}{2}\cdot 6\cdot 2\sqrt{10}=6\sqrt{10}$. Pass back $(\mathbf{6},\mathbf{10})$.

R1-3. Coordinatize the situation. Put vertices $A(0,0)$ and $C(b,b)$ so that the circle has equation $(x-a/2)^2+(y-a/2)^2=a^2/4$. The diagonal $\overline{AC}$ has equation $y=x$, so equate the line and the circle to find the intersection points are at $x=a/2\pm a/4\cdot\sqrt{2}$. Substituting, the larger segment will connect $(3+1.5\sqrt{2},3+1.5\sqrt{2})$ and $(10,10)$. The distance between them can be computed to be $\mathbf{7\sqrt{2}-3}$.

R2-1. The medians meet at the centroid, which is 2/3 of the way to the midpoint of the hypotenuse. Thus, the midpoint of the hypotenuse is $(7.5,3)$. If $P(x,0)$ and $Q(0,y)$, then $\dfrac{x+0}{2}=7.5$ and $\dfrac{y+0}{2}=3$, so the area of the triangle is $\frac{1}{2}\cdot x\cdot y=\frac{1}{2}\cdot 15\cdot 6=\mathbf{45}$.

R2-2. Solving, $\dfrac{1}{\frac{1}{x}+\frac{5}{N}}=12\rightarrow\dfrac{1}{x}+\dfrac{5}{N}=\dfrac{1}{12}\rightarrow x=\dfrac{12N}{N-60}$. Thus, $x=\dfrac{12\cdot 45}{45-60}=12\cdot -3=\mathbf{-36}$.

R2-3. Let $A=x-N$ (an integer). Then $A^2+16=B^2$ implies $16=(B-A)(B+A)$. Factor pairs of 16 are $16\cdot 1$, $8\cdot 2$, and $4\cdot 4$. The $16\cdot 1$ pair implies A and B are fractional and is thus rejected. The $4\cdot 4$ pair implies $A=0$ and $x=N$. The $8\cdot 2$ pair implies $A=3$ and $B=5$, which means $x=N+3$, which is greater than N for all N. Thus, add 3 to the passed number to obtain $-36+3=\mathbf{-33}$.

2017 Tiebreaker Problems

TB-1. The integers from 1 through 2017 are written consecutively to form the 6961-digit number $123456\cdots20162017$. If $S(n)$ denotes the sum of the digits of the number n, compute $S(S(S(S(123456\cdots20162017))))$.

TB-2. A right triangle $\triangle NYS$ has sides such that $NY + YS = 12$ and $NS = 8$. Compute the area of $\triangle NYS$.

2017 Tiebreaker Answers

TB-1. 1

TB-2. $\dfrac{40}{3}$

2017 Tiebreaker Solutions

TB-1. The digit sum of a number is congruent to the number mod 9. The sum of the digits of $A = 123456 \cdots 20162017$ is less than $6961 \cdot 9 < 63000$. This implies $S(A) < 63000$, which means $S(S(A)) < 5 \cdot 9 = 45$. This implies that $S(S(S(A))) \leq 3 + 9 = 12$, and so $S(S(S(S(A)))) \leq 9$. Therefore, find the value of A mod 9, and this will be the answer. Notice that mod 9, $123456 \cdots 20162017 \equiv 1 + 2 + 3 + \cdots + 2 + 0 + 1 + 7$, and also $123456 \cdots 20162017 \equiv 1 + 2 + 3 + \cdots + 2016 + 2017 \equiv \dfrac{2017(2018)}{2} \equiv 2017 \cdot 1009 \equiv 1(1) = \mathbf{1}$.

TB-2. If NY and YS are the legs and NS is the hypotenuse, then $NY^2 + YS^2 = NS^2$ and the area of the triangle is $\frac{NY \cdot YS}{2}$. Notice that, in this case, $(NY + YS)^2 = NY^2 + YS^2 + 2 \cdot NY \cdot YS = NS^2 + 2 \cdot NY \cdot YS$, so $144 = 64 + 2 \cdot NY \cdot YS \to \dfrac{NY \cdot YS}{2} = 20$. However, there is no pair of real numbers a and b for which $a + b = 12$ and $ab = 40$ (think: the equation $x^2 - 12x + 40 = 0$ has a negative discriminant). This means that NS is not the hypotenuse.

Therefore, one of NY and YS is the length of the hypotenuse. Without loss of generality, let NY be the hypotenuse. Then, $(12 - NY)^2 + NS^2 = NY^2 \to 144 - 24NY + NY^2 + NS^2 = NY^2$, so $NY = \dfrac{26}{3}$. This implies that the area of the triangle is $\dfrac{1}{2} \cdot \dfrac{10}{3} \cdot 8 = \mathbf{\dfrac{40}{3}}$.

2018 Contest at SUNY Geneseo (Genesee Valley)

Contents

2018 Team Problems

T-1. Compute the number of values of b for which $x^2 + bx + 2018$ is factorable over the integers.

T-2. Define the *double factorial* of a positive integer n, denoted $n!!$, as follows: $n!! = n(n-2)(n-4)\cdots(3)(1)$ if n is odd, and $n!! = n(n-2)(n-4)\cdots(4)(2)$ if n is even. Compute the odd number N for which $N!!$ has the same number of prime factors (counting multiplicities) as $20!!$.

T-3. Compute the value of $(\sin 40^\circ + \sin 160^\circ + \sin 280^\circ)(\cos 80^\circ + \cos 200^\circ + \cos 320^\circ)$.

T-4. Consider two-digit base-10 positive integers of the form $N = \underline{T}\,\underline{U}$, where $|2T - U| \leq 1$. Compute the sum of all possible primes N.

T-5. Four teams play a single-elimination tournament. The probability that the Aardvarks beat the Burros in a single game is 0.8. The probability that the Chickens beat the Burros in a single game is 0.9, which is the same probability that the Ducks beat the Burros in a single game. The Chickens and Ducks are evenly matched when they play each other. The probability that the Aardvarks beat the Chickens is the same as the probability that the Aardvarks beat the Ducks, and that probability is 0.7. In the first round of the tournament, the Aardvarks play the Burros and the Chickens play the Ducks. The winners of those games play the championship game. Compute the probability that the Chickens win the championship.

T-6. Harriet and Thelma are playing a game with an unfair coin. Harriet goes first. The players take turns tossing the coin until either Harriet tosses heads or Thelma tosses tails. Given that the probability that Harriet wins the game is $\dfrac{10}{19}$, compute the probability that the biased coin comes up heads on any single toss.

T-7. Let $f(z)$ be a function on the complex numbers such that for all z, $f(z) = (f(2018 - z))^2$. Given that $f(100) \neq 0$ and $f(100) \neq 1$, compute the sum of all possible values of $f(100)$.

T-8. An interior diagonal of a convex polyhedron is a segment that connects two vertices of the polyhedron but does not lie on any of the edges or in any of the faces of the polyhedron. Compute the number of interior diagonals of a regular dodecahedron.

T-9. A *Numbrix puzzle* is a 9×9 grid which is filled with the counting numbers from 1 through 81 such that the path between consecutive numbers is either horizontal or vertical – no diagonal paths. In the Numbrix puzzle shown, some numbers are filled in.

5		15		17		29		31
7								37
73								39
71								41
65		63		61		55		43

Compute the sum of the numbers in the three highlighted cells.

T-10. Consider the product $142857 \cdot x$ for various positive integers x. Compute the least positive integer x for which some digit other than the units digit of $142857 \cdot x$ is 0.

2018 Team Answers

T-1. 4

T-2. 35

T-3. 0

T-4. 190

T-5. $\dfrac{21}{100}$ or 0.21

T-6. $\dfrac{2}{5}$ or 0.4 or 40%

T-7. -1

T-8. 100

T-9. 20π

T-10. 37

2018 Team Solutions

T-1. If $x^2 + bx + 2018 = (x+m)(x+n)$, then $m+n = b$ and $mn = 2018$. The only sets of integers $\{m, n\}$ are $\{1, 2018\}$, $\{-1, -2018\}$, $\{2, 1009\}$, and $\{-2, -1009\}$, for a total of **4** values of b.

T-2. Consider that $20!! = 2(2\times2)(2\times3)(2\times2\times2)(2\times5)(2\times2\times3)(2\times7)(2\times2\times2\times2)(2\times3\times3)(2\times 2\times5)$, which has 25 prime factors, counting multiplicities. Now, build an odd double-factorial that has 25 prime factors. Because $N!! = 3(5)(7)(3\times3)(11)(13)(3\times5)(17)(19)(3\times7)(23)(5\times 5)(3\times3\times3)(29)(31)(3\times11)(5\times7)$ has 25 prime factors, $N = \mathbf{35}$.

T-3. Recall that both the real components and the imaginary components of the complex cube roots of 1 add to 0. In other words, $\sin 0^\circ + \sin 120^\circ + \sin 240^\circ = 0$ and $\cos 0^\circ + \cos 120^\circ + \cos 240^\circ = 0$. A rotation of 40° or 80° of those roots around the standard unit circle does not change this fact. As such, both factors equal 0, and so their product is **0**.

T-4. Because N is prime, U is odd. Notice also that $\dfrac{U-1}{2} \le T \le \dfrac{U+1}{2}$, so there are two possible T-values for every U-value. If $U = 1$, $T = 0$ or 1. If $U = 3$, $T = 1$ or 2. If $U = 5$, $T = 2$ or 3. If $U = 7$, $T = 3$ or 4. If $U = 9$, $T = 4$ or 5. The primes generated in this way are 11, 13, 23, 37, 47, and 59, which sum to **190**.

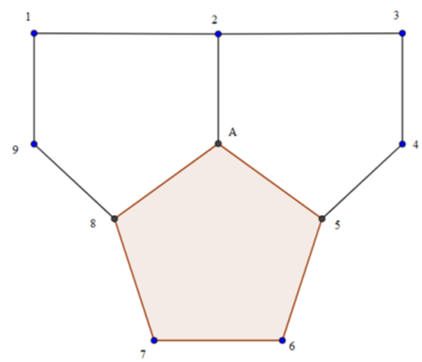
T-5. The probability that the Aardvarks win the championship is $(0.8)(0.7) = 0.56$. The probability that the Burros win the championship is $(0.2)(0.1) = 0.02$. The Chickens and Ducks win the championship with equal probability, so the probability that the Chickens win the championship is $0.5(1 - 0.56 - 0.02) = \mathbf{0.21}$ or $\mathbf{\dfrac{21}{100}}$.

This is a "NYSML Classic". It is very much like question T8 from NYSML2003. Good math problems stand the test of time!

T-6. Let p be the desired probability. The probability that Harriet wins is equal to the probability that she wins on the first toss plus the probability that she wins on a later toss. To win on a later toss, the first toss must be tails, then the second toss must be heads, and after that, the game effectively starts anew. Thus, $\dfrac{10}{19} = p + (1-p)\cdot p \cdot \dfrac{10}{19} \Rightarrow 10 = 19p + (1-p)(p)(10) \Rightarrow 10p^2 - 29p + 10 = 0$. This implies $(5p-2)(2p-5) = 0$, which has only one solution between 0 and 1, so $p = \mathbf{\dfrac{2}{5}}$.

T-7. Notice that $f(100) = (f(1918))^2$, but also $f(1918) = (f(100))^2$, so by substitution, $f(100) = ((f(100))^2)^2 = (f(100))^4$. This implies $(f(100))((f(100))^3 - 1) = 0$. Factoring, this implies $f(100)(f(100) - 1)((f(100))^2 + f(100) + 1) = 0$. Because $f(100) \neq 0$ and $f(100) \neq 1$, it is true that $(f(100))^2 + f(100) + 1 = 0$. By Viete's formulas, the sum of the remaining possible (non-real) values of $f(100)$ is $\dfrac{-1}{1} = \mathbf{-1}$.

T-8. A dodecahedron has twelve pentagonal faces, each of which has five vertices, and each vertex is on three faces, so there are $12 \times 5 \div 3 = 20$ vertices. To draw an interior diagonal, choose one of the 20 vertices, and then select another vertex that is not part of any face containing the first vertex. The diagram shows three adjacent faces of the dodecahedron. From the diagram, observe that out of the 19 vertices other than A, 9 of them share a face with A. Thus, there are 10 possible choices of the second vertex.



Accounting for choosing the vertices in either order, there are $\dfrac{20 \cdot 10}{2} = \mathbf{100}$ interior diagonals.

T-9. Work the top and bottom rows and the leftmost and rightmost columns, first filling in these outermost cells where there is only one choice and then working inward wherever the choice is certain. All unshaded cells were certain.

5	4	**15**	16	**17**		**29**	30	**31**
6	3	14	13				33	32
	2	1	12					**37**
8	9	10	11					38
73	74							**39**
72								40
71	70	69						**41**
66	67	68					45	42
65	64	**63**	62	**61**		**55**	44	**43**

At this point, there are at least two possibilities for each of the remaining numbers, like 75, 18, 28, 60, 56, 46, and 34. Of the two possibilities for 34, one leads immediately to an impossible situation (no place for 36), so place 34 and continue (in red below). Connecting 45 and 55 (in yellow below) as well as 20 through 25 leaves no "holes". Connect 61 and 55 (in green below). Connect 75 to 81 (in blue below).

5	4	**15**	16	**17**	*28*	**29**	30	**31**
	3	14	13	*18*	*27*	*34*	33	32
7	2	1	12	*19*	*26*	*35*	*36*	**37**
8	9	10	11	*20*	*25*	*50*	*49*	38
73	74	*77*	*78*	*21*	*24*	*51*	*48*	**39**
72	*75*	*76*	*79*	*22*	*23*	*52*	*47*	40
71	70	69	*80*	*59*	*58*	*53*	*46*	**41**
66	67	68	*81*	*60*	*57*	*54*	45	42
65	64	**63**	62	**61**	*56*	**55**	44	**43**

If any of these choices had resulted in an impossible situation, then the procedure would be to back up and take another option. That is not the case here, so add the three highlighted cells to obtain $1 + 25 + 80 = \mathbf{106}$.
Query: Can you argue that the solution is unique?

T-10. Notice that 142857 are the repeating digits in the decimal expansion of $\frac{1}{7}$. For the first few multiples of 142857 the digits cycle around as follows:

$142857{\cdot}2 = 285714, 142857{\cdot}3 = 428571, 142857{\cdot}4 = 571428, 142857{\cdot}5 = 714285, 142857{\cdot}6 = 857142.$

However, $142857 \cdot 7 = 999999$. Thinking of 999999 as $1000000 - 1$ for $x = 8$ through $x = 14$, the products are predictable, as follows:

$$142857 \cdot 8 = 1142856,$$

$$142857 \cdot 9 = 1285713,$$

$$142857 \cdot 10 = 1428570,$$

$$142857 \cdot 11 = 1571427,$$

$$142857 \cdot 12 = 1714286,$$

$$142857 \cdot 13 = 1857141,$$

$$142857 \cdot 14 = 1999998.$$

Think of 1999998 as $2000000 - 2$. (In other words, instead of prefixing 1 and subtracting 1 from the rightmost digit of something from the pattern above, prefix 2 and subtract 2 from the rightmost digit of something from the pattern above.) The pattern continues.

Prefixing 2 and subtracting 2 yields $2142855, \ldots, 2857140, 2999997$. This includes all values of x through $x = 21$.

Prefixing 3 and subtracting 3 yields $3142854, \ldots, 3857139, 3999996$. This includes all values of x through $x = 28$.

Prefixing 4 and subtracting 4 yields $4142853, \ldots, 4857138, 4999995$. This includes all values of x through $x = 35$.

Prefixing 5 and subtracting 5 yields $5142852, 5285709, \ldots$.

Therefore, the least value of x for which a nonzero digit appears in a place besides the ones place is $x = \mathbf{37}$.

2018 Individual Problems

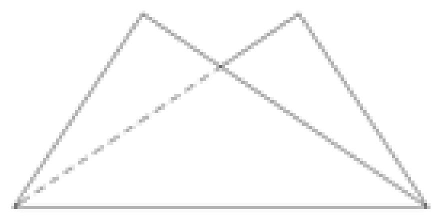
I-1. The sum of the two seven-digit numbers $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}\,\underline{1}\,\underline{7}$ and $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}\,\underline{1}\,\underline{8}$ is 7936835. The five-digit number $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}$ has four not-necessarily-distinct prime factors. Compute the greatest prime factor of the five-digit number $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}$.

I-2. Let r be the greater root of $x^2 + x = 2018$. Compute the value of $(r-1)(r+2)(r-3)(r+4)$.

I-3. If $P(n)$ denotes the product of the digits of $n!$, compute the value of

$$P(0) + P(1) + P(2) + \cdots + P(2017) + P(2018).$$

I-4. A 3-by-4 piece of paper is folded along its diagonal to form a nonconvex polygon as shown.



Compute the area of the polygon in square units.

I-5. Compute the value of $\sqrt{50 \cdot 51 \cdot 54 \cdot 55 + 4}$.

I-6. The numbers $1, 2, 3, \ldots, 100$ are divided into two groups, A and B. Group A contains those numbers whose nearest perfect square is even. For example, 17 is in Group A because it is closer to 16 than to 25. Group B contains those numbers whose nearest perfect square is odd. Let C be the sum of the elements in A and let D be the sum of the elements in B. Compute $C - D$.

I-7. In quadrilateral $ABCD$, $AB = 15$, $AD = 7$, $CB = 20$, and $CD = 24$, and angles B and D are supplementary. Compute $\sin A$.

I-8. Let $f(x) = x^4 - 2x^3 - 23x^2 + 26x + 127$ and let $g(x) = Ax + B$ for some integers A and B. If $f(x) = g(x)$ has exactly two distinct solutions, both of which are real, and if $f(x) < g(x)$ has no real solutions, compute the ordered pair (A, B).

I-9. Compute the least positive integer $a > 2018$ such that $a^3 + 20a^2 + 3a + 18$ is divisible by 17.

I-10. Sally chooses 11 integers at random without replacement from the set of 15 integers $\{1, 2, 3, \ldots, 14, 15\}$. Compute the probability that the median of Sally's 11 integers is 9.

2018 Individual Answers

I-1. 3307

I-2. 4044096

I-3. 18

I-4. $\dfrac{117}{16}$

I-5. 2752

I-6. 100

I-7. $\dfrac{4}{5}$

I-8. $(2, -17)$

I-9. 2022

I-10. $\dfrac{16}{65}$

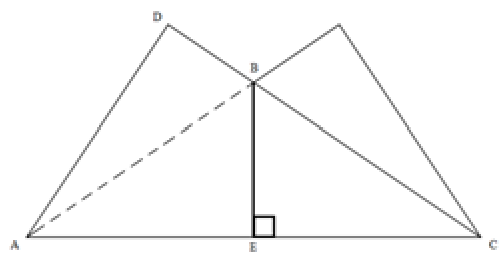
2018 Individual Solutions

I-1. Because $2 \cdot \underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}\,\underline{0}\,\underline{0} = 7936800$, $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L} = 39684$. This is a multiple of 4, so write $39684 = 2^2 \cdot 9921$. Notice that 9921 is a multiple of 3, so $39684 = 2 \cdot 2 \cdot 3 \cdot 3307$. All four of these factors are prime, so the greatest prime factor of $\underline{N}\,\underline{Y}\,\underline{S}\,\underline{M}\,\underline{L}$ is **3307**.

I-2. Multiplying, the desired value is equivalent to $(r^2+r-2)(r^2+r-12) = ((r^2+r-2018)+2016)((r^2+r-2018)+2006)$. Because $r^2+r-2018 = 0$, this is equivalent to $(2016)(2006) = 2011^2 - 5^2 = 4044121 - 25 = \mathbf{4044096}$.

I-3. The product of the digits of $5! = 120$ is 0, as is the product of every factorial greater than $5!$. The first few factorials are $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, and $4! = 24$. Therefore, the desired sum is $P(0)+P(1)+P(2)+P(3)+P(4) = 1+1+2+6+8 = \mathbf{18}$.

I-4. Consider the diagram. Because of overlap, the area of the pentagon equals the area of the original rectangle minus the area of $\triangle ABC$. Because $\overline{AC}$ is the diagonal of the rectangle, $EC = 5/2$ by symmetry. Both $\angle D$ and $\angle BEC$ are right angles, so $\triangle ADC \sim \triangle BEC$ and therefore $\dfrac{EB}{EC} = \dfrac{3}{4} \Rightarrow EB = \dfrac{3}{4} \cdot \dfrac{5}{2} = \dfrac{15}{8}$. Then, $[ABC] = \dfrac{1}{2} \cdot 5 \cdot \dfrac{15}{8} = \dfrac{75}{16}$.



The area of the pentagon is therefore $12 - \dfrac{75}{16} = \mathbf{\dfrac{117}{16}}$.

I-5. Let $x = 50$. Substituting and arranging factors allows the value to be expressed as $\sqrt{x(x+5)(x+1)(x+4)+4}$, which can be rewritten as $\sqrt{(x^2+5x)(x^2+5x+4)+4}$ $= \sqrt{(x^2+5x)^2+4(x^2+5x)+4}$, and this can be expressed as $\sqrt{(x^2+5x+2)^2} = x^2+5x+2$. Now, substituting $x = 50$, the value is $2500+250+2 = \mathbf{2752}$.

This is a "NYSML Classic". It is very much like question I7 from NYSML2013. Good questions are fun in any year!

I-6. The numbers 1 and 2 belong in Group B because they are closer to 1 than to 4. The numbers 3, 4, 5, and 6 belong in Group A because they are closer to 4 than to 9. The numbers 7, 8, 9, 10, 11, and 12 belong in Group B because they are closer to 9 than to 16. Notice that a pattern develops: the next subgroup of numbers (to be sent to Group A) will have two more numbers in it than the previous subgroup, and so on.

Now, consider the difference $C-D = (3+4+5+6+13+14+\ldots+19+20+31+32+\ldots+41+42+57+58+\ldots+71+72+91+92+\ldots+99+100)-(1+2+7+8+\ldots+11+12+21+22+\ldots+29+30+43+44+\ldots+55+56+73+74+\ldots+89+90)$. The first four terms in the first set of parentheses have the same sum as the first four terms in the second set of parentheses, so $C-D = (13+14+\ldots+19+20+31+32+\ldots+41+42+57+58+\ldots+71+72+91+92+\ldots+99+100)-(9+10+11+12+21+22+\ldots+29+30+43+44+\ldots+55+56+73+74+\ldots+89+90)$. In the previous equation, the first eight terms in the first set of parentheses have the same sum as the first eight terms in the second set of parentheses, so $C-D = (31+32+\ldots+41+42+57+58+\ldots+71+72+91+92+\ldots+99+100)-(25+26+\ldots+29+30+43+44+\ldots+55+56+73+74+\ldots+89+90)$. In this equation, the first twelve terms in the first set of parentheses have the same sum as the first twelve terms in the second set of parentheses, so $C-D = (57+58+\ldots+71+72+91+92+\ldots+99+100)-(49+50+\ldots+55+56+73+74+\ldots+89+90)$. In this equation, the first sixteen terms in the first set of parentheses have the same sum as the first sixteen terms in the second set of parentheses, so $C-D = (91+92+\ldots+99+100)-(81+82+\ldots+89+90)$. Now, each set of parentheses has ten terms, and each term in the first set of parentheses is 10 greater than the corresponding term in the second set of parentheses. Therefore, the value of $C-D$ is $10\cdot 10 = \mathbf{100}$.

I-7. By the Law of Cosines, $AC^2 = 7^2+24^2-2\cdot 7\cdot 24\cdot\cos D = 15^2+20^2-2\cdot 15\cdot 20\cdot\cos B$. This implies $625-336\cos D = 625-600\cos B$, which implies $625+336\cos B = 625-600\cos B$, so $\cos B = 0$. This means that angles B and D are both right, and also angles A and C are supplementary. Therefore, the area of $ABCD$ is $\frac{1}{2}\cdot 7\cdot 24+\frac{1}{2}\cdot 15\cdot 20 = 234$. Computing the area differently, $234 = \frac{1}{2}\cdot 7\cdot 15\cdot\sin A+\frac{1}{2}\cdot 20\cdot 24\cdot\sin(180^\circ - A) = \frac{585}{2}\sin A$, so $\sin A = \frac{468}{585} = \mathbf{\frac{4}{5}}$.

I-8. Because $f(x)\geq g(x)$ for all real x, the graph of $y=f(x)$ lies at or above the graph of $y=g(x)$. Because the graphs of f and g intersect in exactly two distinct points and f is never below g, the graph of the line $y=g(x)$ is tangent to the "w-shaped" graph of $y=f(x)$ at two points along the bottom. Let the roots of $f(x)=g(x)$ be r_1 and r_2. Because the graph is tangent to the x-axis at both points, both r_1 and r_2 are double roots of $f(x)=g(x)$. Thus,

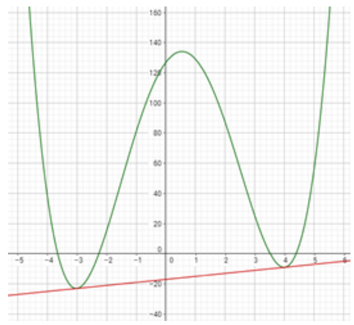
$$f(x)-g(x) = (x-r_1)^2(x-r_2)^2.$$

Either by algebra or by using Vieta's formulas,

$$f(x)-(Ax+B) = x^4-2(r_1+r_2)x^3+(r_1^2+4r_1r_2+r_2^2)x^2-2r_1r_2(r_1+r_2)x+r_1^2r_2^2.$$

Equating coefficients produces the equations $-2(r_1+r_2) = -2 \Rightarrow r_1+r_2 = 1$ and $r_1^2+4r_1r_2+r_2^2 = -23$, but $(r_1+r_2)^2 = 1$, so $1+2r_1r_2 = -23 \Rightarrow r_1r_2 = -12$ and, by inspection,

r_1 and r_2 are 4 and -3. Equating the last two coefficients yields $26 - A = -2r_1r_2(r_1 + r_2) = 24$ which implies $A = 2$ and $127 - B = (r_1r_2)^2 = 144$ which implies $B = -17$. The answer is $(\mathbf{2}, \mathbf{-17})$.
The graphs of f and g are below.



I-9. If $a^3 + 20a^2 + 3a + 18$ is divisible by 17, then $a^3 + 20a^2 + 3a + 18 - (17a^2 + 17) = a^3 + 3a^2 + 3a + 1$ is divisible by 17. Factoring, this implies $(a + 1)^3$ is divisible by 17, and because 17 is prime, 17 divides $a + 1$. The least a greater than 2018 such that $a + 1$ is a multiple of 17 is **2022**.

I-10. For the median to be 9, 9 must be one of the integers. Then, five integers must be greater than 9 (and there are 6 of those in the set) and five must be less than 9 (and there are 8 of those in the set). Thus, the desired probability is $\dfrac{\binom{6}{5} \cdot \binom{8}{5}}{\binom{15}{11}} = \dfrac{\binom{6}{5} \cdot \binom{8}{5}}{\binom{15}{4}} = \dfrac{6 \cdot 56}{1365}$, or $\mathbf{\dfrac{16}{65}}$.

Power Question 2018: Little Kid Triangles

For a nondegenerate triangle in the plane, suppose that its side lengths λ_1, λ_2, and λ_3 satisfy $\lambda_1 \leq \lambda_2 \leq \lambda_3$. If λ_1, λ_2, and λ_3 are natural numbers, we will refer to the triangle as **elementary**, and we may say that it has side lengths $\lambda_1 : \lambda_2 : \lambda_3$ for convenience (since the ratio of the side lengths is what determines the angle measures). Suppose also that the triangle's angle measures (in degrees) δ_1, δ_2, and δ_3 satisfy $\delta_1 \leq \delta_2 \leq \delta_3$. If δ_1, δ_2, and δ_3 are rational numbers, we will refer to the triangle as **rational**. We state that the angle with measure δ_j is opposite the side with length λ_j. Notice that if the three angle measures are known, the side lengths are known up to similarity, and if the side lengths are known, then the angle measures are known. Therefore, we will refer to a triangle $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ as a triangle, even though it defines only the similarity type of a triangle.

We will refer to triangles with $\delta_2 = 60$ or with $\delta_3 = 120$ as **Pythagorish**.

P-1. a. Write the three side lengths of a triangle that is elementary and rational. [3 pts]

b. Write the three angle measures of a triangle that is rational, has a side of length 1, and is not elementary. Also write the other two side lengths of the triangle. [3 pts]

P-2. Show that the triangles Δ_1 with side lengths $5 : 16 : 19$ and Δ_2 with side lengths $5 : 7 : 8$ are Pythagorish. [6 pts]

A valid theorem, which we present without proof, says the following.
Suppose that r_1 and r_2 are rational numbers and neither $r_1 - r_2$ nor $r_1 + r_2$ is a multiple of 180. Then the following two statements are equivalent:
Statement 1: There exists a nontrivial linear combination of the numbers in the set $\{1, \cos r_1^\circ, \cos r_2^\circ\}$ that is equal to 0. Said another way, there exist nonzero integers A, B, and C such that $A + B\cos r_1^\circ + C\cos r_2^\circ = 0$.
Statement 2: Either both numbers r_1 and r_2 are integer multiples of 36, or at least one of them is an integer multiple of 60 or 90.

P-3. Prove that if $\cos r^\circ$ is rational and if r is rational, then r is an integer multiple of 60 or 90. [6 pts]

P-4. Prove that if r is an integer multiple of 60 or 90, then $\cos r^\circ$ is in the set $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. This completes the argument that these five values are the only possible rational values of $\cos r^\circ$. [4 pts]

P-5. Let r_1 and r_2 be rational numbers with $0 < r_1 < 90$ and $0 < r_2 < 90$. Prove that $\dfrac{\cos r_1^\circ}{\cos r_2^\circ}$ is rational if and only if $r_1 = r_2$. [5 pts]

P-6. Now, consider integer solutions (x, y, z) of $x^2 + vxy + y^2 = z^2$ for various integers v. If (x, y, z) solves $x^2 + vxy + y^2 = z^2$ for $v = 0$, then we call the triple a **0-Pythagorean triple**. If the greatest common divisor of the numbers in the set $\{x, y, z\}$ is 1, then we call (x, y, z) a **primitive triple**.

a. If (x, y, z) solves $x^2 + vxy + y^2 = z^2$ for $v = 1$, we call the triple a **(1)-Pythagorean triple**. Find, with justification, any primitive 1-Pythagorean triple. [2 pts]

b. If (x, y, z) solves $x^2 + vxy + y^2 = z^2$ for $v = -1$, we call the triple a **(-1)-Pythagorean triple**. Find, with justification, any primitive (-1)-Pythagorean triple. [2 pts]

P-7. Suppose that $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ is an elementary triangle. In almost all cases, δ_1, δ_2, and δ_3 are transcendental numbers (that is, they are not expressible as the solution to a polynomial equation with integer coefficients). There are three cases where δ_1, δ_2, and δ_3 are not transcendental numbers.

a. Suppose that $\delta_3 = 90$. Show that neither δ_1 nor δ_2 is rational, and that $\lambda_1 : \lambda_2 : \lambda_3 = x : y : z$ for a uniquely determined ordered primitive 0-Pythagorean triple (x, y, z). [2 pts]

b. Suppose that $\delta_3 = 120$. Show that neither δ_1 nor δ_2 is rational, and that $\lambda_1 : \lambda_2 : \lambda_3 = x : y : z$ for a uniquely determined ordered primitive 1-Pythagorean triple (x, y, z). [2 pts]

c. Suppose that $\delta_2 = 60$. Show that neither δ_1 nor δ_3 is rational unless $\delta_1 = \delta_3 = 60$, and that $\lambda_1 : \lambda_2 : \lambda_3 = x : y : z$ for a uniquely determined ordered primitive (-1)-Pythagorean triple (x, y, z). [2 pts]

P-8. It is well known that, given a 0-Pythagorean triple (x, y, z), there exists a unique pair of relatively prime positive integers (m, n) such that $m - n$ is positive and odd and such that $\{x, y\} = \{2mn, m^2 - n^2\}$ and $z = m^2 + n^2$. Show that given a 1-Pythagorean triple (x, y, z), there exists a pair of relatively prime positive integers (m, n) such that $m - n$ is positive and such that $\{x, y\} = \{2mn + n^2, m^2 - n^2\}$ and $z = m^2 + mn + n^2$. [4 pts]

P-9. The previous parts of this Power Question establish that if all of the angle degree-measures and all of the side lengths of a triangle are rational, then the triangle is equilateral. So, let's relax the parameters a bit and see what happens. Call a triangle **elementish** if at least two of the side lengths are natural numbers. Clearly, every elementary triangle is also elementish. Prove that for any rational elementish triangle Δ, either Δ is isosceles or Δ has angle measures of 30°, 60°, and 90°. [5 pts]

P-10. Now, consider triangles whose angle degree-measures are all rational numbers and whose side lengths are all of the form $a\sqrt{b}+c$ for integers a, b, and c where b is not divisible by the square of any prime. There are only fourteen triangles with this property. Some examples have already been discussed. $\Delta_1 = \langle 60, 60, 60 \rangle$ has side lengths in the ratio $1 : 1 : 1$. $\Delta_2 = \langle 30, 30, 120 \rangle$ has side lengths in the ratio $1 : 1 : \sqrt{3}$. $\Delta_3 = \langle 30, 60, 90 \rangle$ has side lengths in the ratio $1 : \sqrt{3} : 2$. List the others in a similar way, naming their angle measures and side lengths. [5 pts]

Solutions to 2018 Power Question

P-1. a. The simplest example is the triangle whose side lengths are all 1 and whose angles all measure 60°. This is not the only such triangle; however, all of the triangles of this type are equilateral.

b. One such triangle has angle measures 30, 60, and 90. In this case, if the smallest side has length 1, the other two sides have lengths $\sqrt{3}$ and 2.

P-2. Use the Law of Cosines. For Δ_1, $19^2 = 5^2 + 16^2 - 2 \cdot 5 \cdot 16 \cdot \cos C$, and solving this yields $\cos C = \dfrac{-1}{2}$, so angle C has measure 120, as needed. Similarly, for Δ_2, it follows that $7^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos B \rightarrow \cos B = \dfrac{1}{2}$, so angle B has measure 60, as needed.

P-3. Assume that r is not an integer multiple of 60 or 90. Choose a rational number q with $0 < q < 1$ such that neither $r - q$ nor $r + q$ is a multiple of 180. It is not possible that both r and q are both integer multiples of 36, and neither r nor q is an integer multiple of 60 or 90, so by the given theorem, there does not exist a nontrivial linear combination of the numbers in the set $\{1, \cos r^\circ, \cos q^\circ\}$ that is equal to 0. Therefore, $\cos r^\circ$ is not rational. This proves the contrapositive of the statement in **P-3**, and so the result is established.

P-4. This result is established by the fact that $f(x) = \cos x^\circ$ is even with period 360. A simple check of cosine values for integer multiples of 60 and 90 in the interval $0 \leq x < 360$ finishes the argument.

P-5. Suppose first that $r_1 \neq r_2$ and that $\dfrac{\cos r_1^\circ}{\cos r_2^\circ}$ is rational. Then neither $r_1 - r_2$ nor $r_1 + r_2$ are multiples of 180, so by the given theorem, either r_1 and r_2 are integer multiples of 36 or one of them is 60. In the first case, either $\dfrac{\cos r_1^\circ}{\cos r_2^\circ}$ or $\dfrac{\cos r_2^\circ}{\cos r_1^\circ}$ equals $\dfrac{\sqrt{5}+1}{\sqrt{5}-1} = \dfrac{\sqrt{5}+3}{2}$, and so $\dfrac{\cos r_1^\circ}{\cos r_2^\circ}$ is irrational, which is a contradiction. In the second case, suppose that it is r_1 that is equal to 60. Then $\cos r_1^\circ = \frac{1}{2}$ is rational, and so $\cos r_2^\circ$ is also rational. By **P-4**, this implies $r_2 = 60$, a contradiction.

The converse is obvious. If $r_1 = r_2$, then $\dfrac{\cos r_1^\circ}{\cos r_2^\circ} = 1$, which is rational.

P-6. Answers will vary, as will the explanations. As an example, $(3, 5, 7)$ is a 1-Pythagorean triple and $(3, 8, 7)$ is a (-1)-Pythagorean triple.

P-7. If the three sides of a triangle are rational, then the cosines of the three angles are rational, which is only possible if the degree measures of the angles are multiples of 60 or 90.

a. If δ_1 or δ_2 were rational, it would require that both are rational. This would imply $\delta_1 = \delta_2 = 60$, which is impossible by the sum of the angle measures of a triangle. Thus, neither δ_1 nor δ_2 is rational. Clearly, $\lambda_1^2 + \lambda_2^2 = \lambda_3^2$ by the Pythagorean Theorem, and dividing each side length by the greatest common factor of the set of side lengths results in a unique primitive 0-Pythagorean triple.

b. If δ_1 or δ_2 were rational, it would require that both are rational. This would imply $\delta_1 = \delta_2 = 60$, which is impossible by the sum of the angle measures of a triangle. Thus, neither δ_1 nor δ_2 is rational. By the Law of Cosines, $\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos 120^\circ = \lambda_3^2$, which implies $\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = \lambda_3^2$, and dividing each side length by the greatest common factor of the set of side lengths results in a unique primitive 1-Pythagorean triple.

c. If δ_1 or δ_3 were rational, it would require that both are rational. This would imply $\delta_1 = \delta_3 = 60$. In this case, $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : 1$, which satisfies the requirement of a (-1)-Pythagorean triple. If $d_1 < 60$, then neither δ_1 nor δ_3 is rational, for reasons similar to the above argument. By the Law of Cosines, $\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos 60^\circ = \lambda_3^2$, which implies $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 = \lambda_3^2$, and dividing each side length by the greatest common factor of the set of side lengths results in a unique primitive (-1)-Pythagorean triple.

P-8. Because (x, y, z) is a 1-Pythagorean triple, it is true that $x^2 + xy + y^2 = z^2$. This implies $\left(\frac{x}{z}\right)^2 + \frac{x}{z} \cdot \frac{y}{z} + \left(\frac{y}{z}\right)^2 = 1$. There exist relatively prime integers m and n such that $\frac{y+z}{x} = \frac{m}{n}$. Notice that $m > n$. For these values of m and n, it is true that $(m^2 - n^2)^2 + (2mn + n^2)(m^2 - n^2) + (2mn + n^2)^2 = (m^4 - 2m^2n^2 + n^4) + (2m^3n - 2mn^3 + m^2n^2 - n^4) + (4m^2n^2 + 4mn^3 + n^4) = m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4 = (m^2 + mn + n^2)^2$, as desired.

P-9. Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be a triangle with side lengths in the ratio $\lambda_1 : \lambda_2 : \lambda_3$. Suppose that Δ is rational and elementish. Therefore, two of the side lengths ℓ_m and ℓ_n are rational. By the Law of Sines, $\frac{\ell_m}{\ell_n} = \frac{\sin \delta_m^\circ}{\sin \delta_n^\circ}$, so $\frac{\cos(90 - \delta_m)^\circ}{\cos(90 - \delta_n)^\circ}$ is rational. Assume without loss of generality that $\delta_m \geq \delta_n$. This implies that $\delta_n < 90$. Proceed by cases.
If $\delta_m > 90$, then by **P-5**, $\delta_m - 90 = 90 - \delta_n$, and this implies $\delta_m + \delta_n = 180$, which is impossible.
If $\delta_m = 90$, then by **P-3**, $90 - \delta_n = 60$, so the triangle has angle measures of 30°, 60°, and 90°.
If $\delta_m < 90$, then by **P-5**, $\delta_m = \delta_n$, and the triangle is isosceles. This completes the proof.

P-10. The rest of the list is as follows. It can be confirmed by arithmetic and trigonometry.
$\Delta_4 = \langle 15, 15, 150 \rangle$ has side lengths in the ratio $\sqrt{2} : \sqrt{2} : \sqrt{3} + 1$.
$\Delta_5 = \langle 30, 75, 75 \rangle$ has side lengths in the ratio $\sqrt{3} - 1 : \sqrt{2} : \sqrt{2}$.
$\Delta_6 = \langle 36, 36, 108 \rangle$ has side lengths in the ratio $2 : 2 : \sqrt{5} + 1$.
$\Delta_7 = \langle 36, 72, 72 \rangle$ has side lengths in the ratio $\sqrt{5} - 1 : 2 : 2$.
$\Delta_8 = \langle 45, 45, 90 \rangle$ has side lengths in the ratio $1 : 1 : \sqrt{2}$.
$\Delta_9 = \langle 15, 30, 135 \rangle$ has side lengths in the ratio $\sqrt{3} - 1 : \sqrt{2} : 2$.
$\Delta_{10} = \langle 15, 45, 120 \rangle$ has side lengths in the ratio $\sqrt{3} - 1 : 2 : \sqrt{6}$.
$\Delta_{11} = \langle 15, 60, 105 \rangle$ has side lengths in the ratio $\sqrt{3} - 1 : \sqrt{6} : \sqrt{3} + 1$.
$\Delta_{12} = \langle 15, 75, 90 \rangle$ has side lengths in the ratio $\sqrt{3} - 1 : \sqrt{3} + 1 : 2\sqrt{2}$.
$\Delta_{13} = \langle 30, 45, 105 \rangle$ has side lengths in the ratio $\sqrt{2} : 2 : \sqrt{3} + 1$.
$\Delta_{14} = \langle 45, 60, 75 \rangle$ has side lengths in the ratio $2 : \sqrt{6} : \sqrt{3} + 1$.

Author's Note: This Power Question was inspired by the article More Grade School Triangles *in the April 2017 edition of the* American Mathematican Monthly*, published by the MAA. The author of the article is Arno Berger.*

2018 Relay Problems

R1-1. Some four-digit numbers of the form $\underline{2}\,\underline{0}\,\underline{A}\,\underline{8}$ are divisible by 12. Compute the sum of all possible values of A such that $\underline{2}\,\underline{0}\,\underline{A}\,\underline{8}$ is divisible by 12.

R1-2. *Let N be the number you will receive.* Compute the number of ordered pairs of positive integers (x, y) with $x < y$ for which $\dfrac{xy}{x+y} = N$.

R1-3. *Let N be the number you will receive.* Compute the number of lattice points in the interior of the graph of $|x| + |y| < N$.

R2-1. Suppose that n leaves a remainder of 24 when divided by 77. Given that n leaves a remainder of A when divided by 7 and a remainder of B when divided by 11, compute $A + B$.

R2-2. *Let N be the number you will receive.* Suppose that $a = \log_p q$ and $b = \log_q p$. Given that $a + b = N$, compute $a^2 + b^2$.

R2-3. *Let N be the number you will receive.* Two concentric circles have radii R and r. The annulus between the circles is divided into N regions of equal area by lines that pass through the common center of both circles. The area of one of the N regions of the annulus is 2018π. Given that $R - r = N$, compute r.

2018 Relay Answers

R1-1. 10

R1-2. 4

R1-3. 25

R2-1. 5

R2-2. 23

R2-3. $\frac{1995}{2}$ or $997\frac{1}{2}$ or 997.5

2018 Relay Solutions

R1-1. To be divisible by 12, the number must be divisible by 3 and 4. Applying the divisibility test for 4, the two-digit number $\underline{A}\,\underline{8}$ must be a multiple of 4, so A must be even. Applying the divisibility test for 3, the sum $10 + A$ must be divisible by 3, so A is 2 or 5 or 8. The values of A that satisfy both divisibility conditions are 2 and 8, which sum to **10**.

R1-2. Solving for y in terms of x, $xy = Nx + Ny \Rightarrow y = \dfrac{Nx}{x-N} = \dfrac{Nx - N^2}{x-N} + \dfrac{N^2}{x-N} = N + \dfrac{N^2}{x-N}$. Therefore, if y is to be an integer, $(x - N)$ must be a factor of N^2. Substituting, $(x - 10)$ must be a factor of 100. Therefore, x is in the set $\{11, 12, 14, 15, 20, 30, 35, 60, 110\}$. Solving for y reveals that the greatest values are paired with the least to form ordered pairs (x, y). The ordered pairs are $(11, 110)$, $(12, 60)$, $(14, 35)$, $(15, 30)$, $(20, 20)$ (and this last one is rejected). There are **4** ordered pairs.

R1-3. For $N = 1$, there is only the origin in the set of lattice points.
For $N = 2$, there are three lattice points on the y-axis and the points $(\pm 1, 0)$, a total of 5.
For $N = 3$, there are five lattice points on the y-axis, three at each of $x = \pm 1$, and the points $(\pm 2, 0)$, a total of 13.
Notice that the number of lattice points goes up by multiples of 4. You are now ready to predict, and you can do so for the passed value of 4. The answer is $13 + 12 = \mathbf{25}$.

R2-1. Notice that n is of the form $77k + 24$. When n is divided by 7, the 7 divides $77k$, and the 24 leaves a remainder of 3 when divided by 7, so $A = 3$. Similarly, 24 leaves a remainder of 2 when divided by 11, so $B = 2$. The answer is $A + B = \mathbf{5}$.

R2-2. By algebra, $(a+b)^2 = a^2+b^2+2ab = N^2 \Rightarrow a^2+b^2 = N^2-2ab$. Notice that $ab = \dfrac{\ln p}{\ln q} \cdot \dfrac{\ln q}{\ln p} = 1$, so the answer is $N^2 - 2$. Substituting, the value to pass is **23**.

R2-3. The area of one section of the annulus is $\dfrac{\pi}{N}(R^2 - r^2)$, so $\pi(R + r)(R - r) = 2018 \cdot N \cdot \pi$. Because $R - r = N$, this implies $R + r = 2018$. Adding the two linear equations together, $2R = 2018 + N \to R = \dfrac{2018 + N}{2}$, so $r = \dfrac{2018 + N}{2} - N = \dfrac{2018 - N}{2} = \mathbf{\dfrac{1995}{2}}$.

2018 Tiebreaker Problems

TB-1. Compute the sum of all positive integers n for which $n^2 + 3n + 2018$ is a perfect square.

TB-2. Given $\triangle ABC$ with D on $\overline{AB}$ and E on $\overline{AC}$ such that $\overline{CD}$ bisects $\angle ACB$ and $\overline{BE}$ bisects $\angle ABC$. Given that $AD = 3$, $AE = 4$, and $EC = 8$, compute BC.

2018 Tiebreaker Answers

TB-1. 2193

TB-2. 12

2018 Tiebreaker Solutions

TB-1. Because $n^2+3n+2018$ is a perfect square, $n^2+3n+2018=(n+d)^2$ for some positive integer d. By algebra, $n^2+3n+2018=n^2+2nd+d^2 \to 2018-d^2=n(2d-3)$. This implies that $d^2<2018<2025$, so $d<45$. Because $(2d-3)$ is a factor of $2018-d^2$, it must also be a factor of $8072-4d^2$. Notice also that $(2d-3)$ is a factor of $(2d-3)(2d+3)=4d^2-9$, so $(2d-3)$ is a factor of $(8072-4d^2)+(4d^2-9)=8072-9=8063$. Because $8063=11\cdot 733$, and because 11 and 733 are prime, $(2d-3)$ has the value 1 or 11 or 733 or 8063. The latter two values contradict the fact that $d<45$, so solve $2d-3=1$ to obtain $d=2$ and $2d-3=11$ to obtain $d=7$. If $d=2$, then $2018-2^2=n\cdot 1 \to n=2014$. If $d=7$, then $2018-7^2=n\cdot 11 \to n=179$. The sum of the two possible positive integer values of n is $2014+179=\mathbf{2193}$.

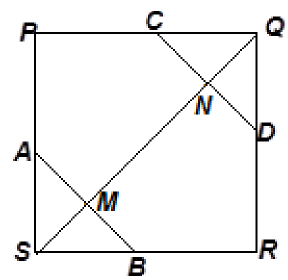
TB-2. Because $\overline{BE}$ is an angle bisector, it splits the side of the triangle to which it is drawn in the ratio of the other two sides. Thus, $AB=4x$ and $BC=8x$ for some number x. Similarly, $AD=12y$ and $DB=8xy$ for some number y. Because $AD=3$, $12y=3 \to y=\dfrac{1}{4}$, and so $DB=8x\cdot\dfrac{1}{4}=2x$. Because $AD+DB=AB$, $3+2x=4x$, which implies $2x=3$, so $BC=8x=\mathbf{12}$.

2019 Contest at Middletown High School (DUSO)

Contents

2019 Team Problems

T-1. In the figure, $PQRS$ is a unit square. Points A, B, C, and D are on $\overline{PS}$, $\overline{SR}$, $\overline{PQ}$, and $\overline{QR}$, respectively such that $AS = BS = CQ = DQ < \frac{1}{2}$. Points M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Given that $MN = 1$, compute AB.



T-2. Given that a and b are integers with $1 \leq a \leq 2019$ and $1 < b \leq 2019$, compute the number of solutions (a, b) to

$$\log_b \left(a^{\log_b a}\right) = \log_b a.$$

T-3. Square $ABCD$ has side length 8. A circle passes through A and B, and is tangent to side $\overline{CD}$. Compute the radius of the circle.

T-4. A state has ten cities: Allenberg, Bergville, Centerville, Downtown, Ellocity, Funkytown, Gutenberg, Halotown, Ipswich, and Johnsonville. Every city except Johnsonville is connected to every other city except Johnsonville by two separate roads. Johnsonville is on an island, and is only connected to Allenberg and Bergville. A couple on their honeymoon decides to tour the state, starting at Centerville, travelling to every city exactly once, and ending at Downtown. Compute the number of distinct ways they can take their trip. *Note that only the order in which the cities are visited matters to determine a trip.*

T-5. Richard is reading a 1225-page novel. On the first day, he reads through page x of the novel, completing exactly $m\%$ of the novel, where m is a two-digit number. The next day, he reads through page y of the novel. After the two days of reading, Richard completes exactly $n\%$ of the novel, where n is the two-digit integer formed by reversing the digits of m. Compute $x + y$.

T-6. Compute the number of nine-digit numbers $\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C}$, with $A \neq 0$ and with A, B, and C not necessarily distinct, that are divisible by 27.

T-7. Let S be the set of points equidistant from the point $F(1,2)$ and the line $2x + y = 1$. The domain of the relation S is $\{x \mid x \geq k\}$. Compute k.

T-8. The number 2019 has a digit sum of 12, and the number 2019 is not divisible by 12. Compute the number of four-digit integers N with $1000 \leq N \leq 9999$ for which N has a digit sum of 12 and for which N is not divisible by 12.

T-9. The graph of $y = -\tan(2x + A)$ has an asymptote at $x = \dfrac{\pi}{5}$ for various values of A. Let the increasing sequence a_1, a_2, a_3, $\ldots$ denote the positive values of A for which the graph of $y = -\tan(2x + A)$ has an asymptote at $x = \dfrac{\pi}{5}$. Compute a_{2019}.

T-10. When two different cubic polynomials, a quartic polynomial, the line $y = 0$, and the line $x = 0$ are all graphed in the Cartesian coordinate plane, the graphs divide the plane into different regions, some bounded and some unbounded. Let N be the number of regions formed by the graphs. Compute the sum of all possible N.

2019 Team Answers

T-1. $\sqrt{2} - 1$

T-2. 4036

T-3. 5

T-4. 1440

T-5. 1617

T-6. 100

T-7. $\dfrac{1}{4}$

T-8. 248

T-9. $\dfrac{20181\pi}{10}$

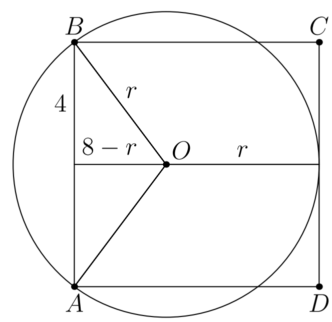
T-10. 475

2019 Team Solutions

T-1. By the Pythagorean Theorem, $QS = \sqrt{2}$. Because $\triangle ASM$ is isosceles and right, $SM = \dfrac{AB}{2}$. Similarly, $NQ = \dfrac{AB}{2}$. Thus, $QS = \sqrt{2} = SM + MN + NQ = AB + 1$ and $AB = \mathbf{\sqrt{2} - 1}$.

T-2. Using the power rule of logarithms, the given equation implies $\log_b a \cdot \log_b a = \log_b a$, which has solutions if $\log_b a = 0$ or $\log_b a = 1$. The first equation has a solution if and only if $a = 1$, and so any pair of the form $(1, b)$ is a solution. There are 2018 solutions of this type. The second equation has solution if and only if $a = b$, and so any pair of the form (b, b) is a solution. There are 2018 solutions of this type. None of the solutions of the type $(1, b)$ are also solutions of the type (b, b) because $b \neq 1$. Thus the number of solutions is $2018 + 2018 = \mathbf{4036}$.

T-3. Consider the following diagram.



Let the radius of the circle be r, and let its center be O. Then the distance from O to $\overline{CD}$ is r and the distance from O to $\overline{AB}$ is $8 - r$. The foot of the altitude from O to $\overline{AB}$ splits $\triangle OAB$ into two right triangles, each with legs of length 4 and $8 - r$, and each with hypotenuse r. Therefore, from the Pythagorean Theorem,

$$4^2 + (8 - r)^2 = r^2 \Longrightarrow 80 - 16r + r^2 = r^2 \Longrightarrow 80 = 16r \Longrightarrow r = \mathbf{5}.$$

Alternate Solution: Let the point of tangency of the circle with $\overline{CD}$ be E. Note that triangle ABE has area equal to $\frac{AB \cdot BE \cdot EA}{4r}$, while also having base $AB = 8$ and height 8 and thus its area is $\frac{8 \cdot 8}{2} = 32$; set these two quantities equal to solve for r.

T-4. The trip can be thought of as an ordering of the letters A through J, with three conditions: C must be at the start, D must be at the end, and A, J, B or B, J, A must appear somewhere. Each of these arrangements corresponds to a unique honeymoon trip, and each trip corresponds to a letter ordering, so there are the same number of letter orderings as trips. It suffices to count the number of letter orderings. To do this, choose where city J will land in the ordering, then how A and B will be oriented about J, and then how the remaining letters will be arranged. There are six ways to choose J's position, two ways to choose the orientation of A and B, and 5! ways to order the remaining letters. Thus there are $6 \cdot 2 \cdot 5! = 12 \cdot 120 = \mathbf{1440}$ total possible honeymoon trips.

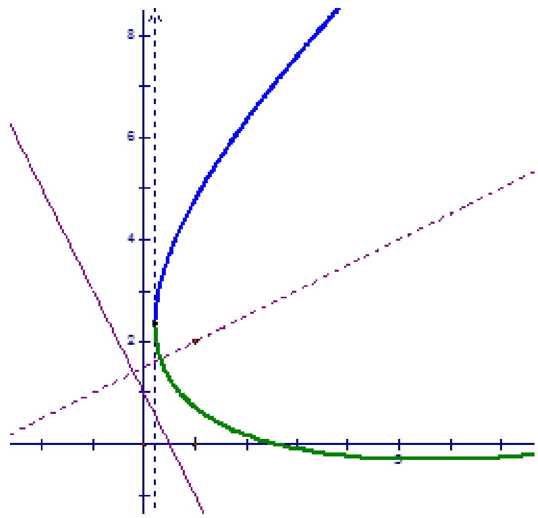
T-5. Because $1225 = 25 \cdot 49 = 5^2 \cdot 7^2$ and m and n are two-digit integers, $\frac{m}{100}$ and $\frac{n}{100}$ in reduced form must be of the form $\frac{k}{25}$ for some integer k. Possible values of k are 3, 4, $\ldots$, 24. So possible values of m and n are 12, 16, $\ldots$, 96. After examining the possible values of m and n, the only values of m and n for which the one is formed by reversing the digits of the other are 48 and 84. Thus $x+y = \frac{48}{100} \cdot 1225 + \frac{84}{100} \cdot 1225 = \frac{132}{100} \cdot 5^2 \cdot 7^2 = \frac{33}{25} \cdot 25 \cdot 49 = 33 \cdot 49 = \mathbf{1617}$.

T-6. Note that $10^3 - 1 = 999 = 27 \cdot 37$. Therefore $10^3 \equiv 1 \bmod 27$. As a result,

$$\begin{aligned}
\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C} &= A(10^8 + 10^5 + 10^2) + B(10^7 + 10^4 + 10) + C(10^6 + 10^3 + 1) \\
&\equiv A(10^2 + 10^2 + 10^2) + B(10 + 10 + 10) + C(1 + 1 + 1) \bmod 27 \\
&\equiv 3 \cdot \underline{A}\,\underline{B}\,\underline{C} \bmod 27.
\end{aligned}$$

This means that $\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C}\,\underline{A}\,\underline{B}\,\underline{C}$ is divisible by 27 if and only if $3 \cdot \underline{A}\,\underline{B}\,\underline{C}$ is divisible by 27 (or, alternatively, if $\underline{A}\,\underline{B}\,\underline{C}$ is divisible by 9). It therefore suffices to find the number of three-digit numbers $\underline{A}\,\underline{B}\,\underline{C}$, with $A \neq 0$, that are divisible by 9. The greatest three-digit multiple of 9 is $999 = 9 \cdot 111$, and the least is $108 = 9 \cdot 12$, so there are $111 - 12 + 1 = 100$ such numbers. Therefore there are **100** nine-digit numbers of the desired form.

T-7. The set of points S is a parabola whose axis of symmetry is not parallel to either of the coordinate axes. Let an arbitrary point P in S have coordinates (x, y). Using the formulas for distances point-to-point and point-to-line, the coordinates of P satisfy $\sqrt{(x-1)^2 + (y-2)^2} = \frac{|2x + y - 1|}{\sqrt{2^2 + 1^2}}$. Squaring both sides, $(x-1)^2 + (y-2)^2 = \frac{(2x + y - 1)^2}{5}$, which implies $5(x^2 - 2x + 1 + y^2 - 4y + 4) = 4x^2 + y^2 + 1 + 4xy - 4x - 2y$, which implies $x^2 - 4xy + 4y^2 - 6x - 18y + 24 = 0$. Writing this as a quadratic equation in y, this is equivalent to $4y^2 - (4x + 18)y + (x^2 - 6x + 24) = 0$. Using the quadratic formula to solve for y in terms of x yields $y = \frac{(4x + 18) \pm \sqrt{(4x + 18)^2 - 16(x^2 - 6x + 24)}}{8} = \frac{(2x + 9) \pm \sqrt{60x - 15}}{4}$. Thus the values of x in the domain are precisely the solutions to $60x - 15 \geq 0 \to x \geq \frac{1}{4}$, so $k = \mathbf{\frac{1}{4}}$. The graph is shown in the figure.



T-8. First, compute the number of four-digit integers that have a digit sum of 12. Then compute the number of those that are multiples of 12.

Use the "balls and urns" method to count the number of four-digit integers whose digit sum is 12. That is, imagine that there are 12 balls in a line, and some dividers are inserted between the balls to assign them into distinct groups ("urns"). (More information on this appears at `https://artofproblemsolving.com/wiki/index.php/Ball-and-urn`, among other places.) To divide the 12 balls into 4 urns requires 3 dividers, so the number of ways to break 12 balls into 4 urns is $\binom{12+3}{3} = \binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 5 \cdot 7 \cdot 13 = 5 \cdot 91 = 455$. However, not all of these are possible four-digit numbers in base 10. For example, (12) is not a digit in base 10, so the 4 numbers (12)000, 0(12)00, 00(12)0, and 000(12) must be discounted. So, too, must all numbers with (11) as a digit. There are 4 places to put the (11) and 3 remaining places to put the 1 that makes the digit sum 12, so there are $4 \cdot 3 = 12$ four-digit numbers with (11) as a digit, and these are discounted. Also, all numbers with (10) as a digit must be discounted. These numbers come in two kinds: (10)200 (and all its permutations) and (10)110 (and all its permutations). In each case, there are $4 \cdot 3 = 12$ of them, and all $12 + 12 = 24$ four-digit numbers are discounted. Lastly, discount all numbers with a 0 in the thousands place. Using "balls and urns" again, there are $\binom{12+2}{2} = \binom{14}{2} = \frac{14 \cdot 13}{2 \cdot 1} = 7 \cdot 13 = 91$ of these, but $3 + 6 + 6 + 3 = 18$ of these have been discounted already in previous cases, so there are $91 - 18 = 73$ more to discount. Thus there are $455 - 4 - 12 - 24 - 73 = 342$ four-digit numbers with a digit sum of 12.

A number is divisible by 12 if it is divisible by 3 and 4. The digit sum of 12 assures that the numbers with digit sum 12 are divisible by 3, so count the number of numbers that are divisible by 4. The last two digits are relevant here, and a careful accounting reveals the following:

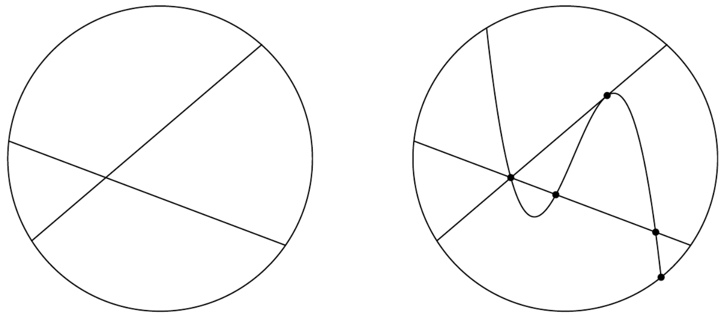
Last 2 dig	Numbers divisible by 4	Count	Last 2 dig	Numbers divisible by 4	Count
00	$3900, 4800, \ldots, 9300$	7	52	$1452, 2352, \ldots, 5052$	5
04	$1704, 2604, \ldots, 8004$	8	56	1056	1
08	$1308, 2208, \ldots, 4008$	4	60	$1560, 2460, \ldots, 6060$	6
12	$1812, 2712, \ldots, 9012$	9	64	$1164, 2064$	2
16	$1416, 2316, \ldots, 5016$	5	68	None	0
20	$1920, 2820, \ldots, 9120$	9	72	$1272, 2172, 3072$	3
24	$1524, 2424, \ldots, 6024$	6	76	None	0
28	$1128, 2028$	2	80	$1380, 2280, \ldots, 4080$	4
32	$1632, 2532, \ldots, 7032$	7	84	None	0
36	$1236, 2136, 3036$	3	88	None	0
40	$1740, 2640, \ldots, 8040$	8	92	1092	1
44	$1344, 2244, \ldots, 4044$	4	96	None	0
48	None	0			

There are 94 multiples of 12 with a digit sum of 12, so the answer is $342 - 94 =$ **248** four-digit numbers that have a digit sum of 12 and that are not multiples of 12.

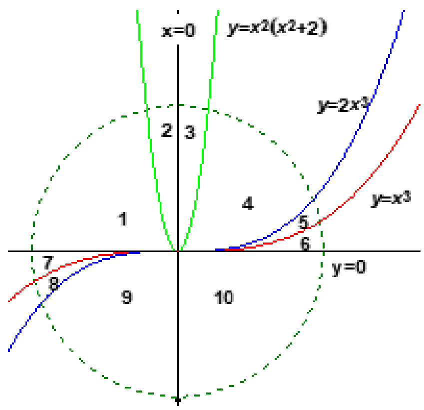
This question is similar to Question T-8 from NYSML2009. The original question was written by Dr. Leo Schneider, who authored NYSML from 2001 to 2010. We use the question here to honor his memory.

T-9. The period of the graph of $y = -\tan\theta$ is π, so the period of $y = -\tan(2x + A)$ is $\pi/2$. The asymptotes of $y = -\tan(2x + A)$ occur at multiples of $\pi/2$ away from $x = K$ where K is the least positive number for which $x = K$ is an asymptote. Set $2x + A = \dfrac{\pi}{2} + n \cdot \pi$. Then set $x = \pi/5$ to obtain that $A = \pi/10$ if $n = 0$, and this is the least positive number for which there is an asymptote so $K = \pi/10$ and $a_{2019} = \pi/10 + 2018\pi = \boldsymbol{\dfrac{20181\pi}{10}}$.

T-10. If an infinite, unbounded, non-self-intersecting curve is drawn in the plane, then it splits the plane into two regions. If there are k such curves with none intersecting, then they split the plane into $k + 1$ regions. If an additional curve is drawn and it intersects the other curves at i points, then this curve will increase the number of regions by $i + 1$: one region by the very introduction of another curve, and i regions from the intersections. This is true when two curves intersect at a single point but also when the points of intersection are tangency points and at points where more than two curves intersect. This can be seen by enclosing these i points in a circle and seeing how many regions inside the circle are formed when the additional curve is drawn. Because all of the intersection points are contained inside the circle, the number of regions inside the circle will be equal to the number of regions if the circle is removed. An example is shown below: a cubic that runs from the left to the right creates five new regions when it hits the dots shown, where the fifth dot represents the existing cubic making a fifth region.



Because the x-axis and the y-axis intersect at $(0, 0)$, it is guaranteed that there is at least one point of intersection. It is possible to draw two distinct cubic functions and a quartic that also intersect at only $(0, 0)$. A diagram is shown below, and it motivates the understanding that the minimum number of regions is 10.



Therefore if k curves are drawn that intersect in i places, they create $1+k+i$ regions. As k is fixed at 5 as per the problem statement, find what the possible values of i are. To minimize i, find the least possible number of intersection points. There is one guaranteed intersection: $(0, 0)$, where the x-axis and y-axis meet. To have this be the only intersection point, choosing the cubics $y = x^3$ and $y = 2x^3$ and the quartic $y = x^2(x^2 + 100)$ ensures that the polynomials all intersect the x-axis at only $(0, 0)$, they all pass the y-axis at $(0, 0)$, and they don't intersect each other anywhere else. Thus the minimum possible value of i is 1.

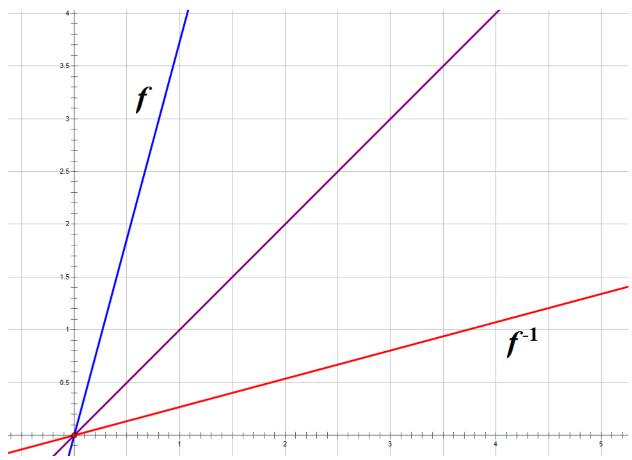
The maximum value i is attained when the cubics, the quartic, and the lines intersect in as many places as possible. The y-axis is guaranteed to intersect the other curves exactly four times, and the rest of the intersection points must come from the other curves. Note that the other four curves are all polynomial curves: one degree four polynomial, two degree three polynomials, and one constant polynomial ($y = 0$, the x-axis). If two polynomials $f(x)$ and $g(x)$ have degrees m and n respectively, then their intersections are when $f(x) - g(x) = 0$, and the polynomial $f(x) - g(x)$ can have degree anywhere between 0 and $\max(m, n)$, so there are anywhere between 0 and $\max(m, n)$ points of intersection between f and g. Therefore the three non-constant polynomials have up to $3 + 3 + 4 = 10$ intersections with the x-axis, the cubics have up to 3 intersections with each other, and the quartic has up to $4 + 4 = 8$

intersections with the two cubics. Therefore there are at most $10 + 3 + 8 = 21$ intersections among the polynomial curves, plus 4 from the y-axis intersections, giving a maximum value of $21 + 4 = 25$ for i. Note that the above argument demonstrates that i can be any number from 1 to 25 inclusive, so $N = 1 + k + i = 6 + i$ can be anywhere from 7 to 31 inclusive. There are $31 - 7 + 1 = 25$ possible values for N, and the sum of all these values is $25 \cdot \frac{10+31}{2} = \mathbf{475}$.

2019 Individual Problems

I-1. For two positive numbers a and b, with $a > b$, their quotient, difference, and sum are in the respective ratio $20 : 1 : 9$. Compute a.

I-2. The graphs of a linear function $y = f(x)$, its inverse $y = f^{-1}(x)$, and the angle bisector of the acute angle between the graphs are shown in the diagram below. All three lines pass through the origin. Given that the sum of the slopes of the three lines is 5, compute the sum of the squares of the slopes of the three lines.



I-3. The number 157751 is a palindrome because it reads the same forward and backward. The odometer of a car shows 157751 miles at the beginning of a trip. At the end of the trip, the odometer reading is the next greater palindrome. Compute the number of miles in the trip.

I-4. Compute the positive integer k for which there are exactly 2019 integer values of x such that $\sqrt{(k+x)(k-x)-4(1-x)}$ is a real number.

I-5. Let the sequence $\{a_n\}$ be defined by $a_1 = 64$ and $a_n = a_{n-1} + 8$ for $n \geq 2$. Compute the product $\left(\log_{a_1} a_2\right)\left(\log_{a_2} a_3\right)\left(\log_{a_3} a_4\right)\cdots\left(\log_{a_{120}} a_{121}\right)$.

I-6. Let m and n be the roots of $x^2 - 20x - 19 = 0$ with $m < n$. Compute

$$2m^2 + 5mn - n^2 - 20m + 40n + 19.$$

I-7. An isosceles trapezoid has diagonals of length 13 and legs of length 8. Compute the product of the lengths of its bases.

I-8. A sequence of numbers $\{a_n\}$ is determined as follows. Let $a_0 = 0$. For every $n \geq 1$, determine a_n by flipping a fair coin. If the coin comes up heads, then $a_n = a_{n-1} + 1$. If the coin comes up tails, then $a_n = a_{n-1} - 1$. Given that $a_{10} = 0$, compute the probability that $a_i = 1$ and $a_j = -1$ for some i and j each between 0 and 10.

I-9. The number $2019 = 1 + 169 + 1849$ can be expressed as the sum of three odd perfect squares. Compute the least integer N with $N > 2019$ that can be expressed as the sum of three odd perfect squares.

I-10. In $\triangle ABC$, $AB = 13$, $BC = 14$, and $AC = 15$. Point D is on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle A$ and point E is on $\overrightarrow{AD}$ such that $\overline{CE} \perp \overrightarrow{AD}$. Compute the area of $\triangle BEA$.

2019 Individual Answers

I-1. $\dfrac{5}{16}$ or 0.3125

I-2. 15

I-3. 1100

I-4. 1009

I-5. $\dfrac{5}{3}$ or $1\dfrac{2}{3}$ or $1.\overline{6}$

I-6. 343

I-7. 105

I-8. $\dfrac{2}{3}$ or $0.\overline{6}$

I-9. 2027

I-10. 42

2019 Individual Solutions

I-1. For some value of k, it is true that $\dfrac{a}{b} = 20k$, $a - b = k$, and $a + b = 9k$. Adding the last two equations yields $2a = 10k \to a = 5k$. Substituting, $\dfrac{5k}{b} = 20k \to b = \dfrac{1}{4}$. Substituting again, $5k - \dfrac{1}{4} = k \to k = \dfrac{1}{16} \to a = 5 \cdot \dfrac{1}{16} = \mathbf{\dfrac{5}{16}}$.

I-2. Let $f(x) = mx$; then $f^{-1}(x) = \dfrac{1}{m}x$. The graph of f^{-1} is obtained by reflecting f over $y = x$, so the angle bisector of the acute angle is $y = x$. The sum of the slopes of the three lines is $m + 1 + \dfrac{1}{m} = 5$, which implies $m + \dfrac{1}{m} = 4$. Squaring both sides yields $m^2 + 2 + \dfrac{1}{m^2} = 16$. Now notice that the answer to the question is the value of $m^2 + 1 + \dfrac{1}{m^2}$, so subtract 1 to obtain $16 - 1 = \mathbf{15}$.

I-3. The next greater palindrome would begin 15 and must have the next digit greater than 7 in the hundreds place, so the next greater palindrome is 158851. The car's trip was therefore $158851 - 157751 = \mathbf{1100}$ miles.

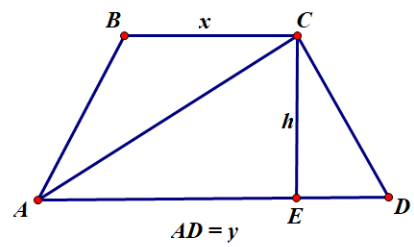
I-4. Simplify the radicand to obtain $\sqrt{(k+x)(k-x) - 4(1-x)} = \sqrt{k^2 - (x^2 - 4x + 4)}$ which is equivalent to $\sqrt{k^2 - (x-2)^2}$. Because the radicand is nonnegative, $(x-2)^2 \leq k^2$ which implies $-k \leq x - 2 \leq k$ and also $2 - k \leq x \leq k + 2$. The number of integers between $2 - k$ and $k + 2$ inclusive is $k + 2 - (2 - k) + 1 = 2k + 1$. Equating this to 2019 and solving $2k + 1 = 2019$ yields $k = \mathbf{1009}$.

I-5. Notice that the sequence $\{a_n\}$ is arithmetic with a common difference of 8. Thus it follows that $a_{121} = 64 + 8 \cdot 120 = 1024$. The product is equal to $(\log_{64} 72)(\log_{72} 80)(\log_{80} 88) \cdots (\log_{1016} 1024)$, or $\log_{64} 1024$, which is equal to $\dfrac{\log 72}{\log 64} \cdot \dfrac{\log 80}{\log 72} \cdot \dfrac{\log 88}{\log 80} \cdot \ldots \cdot \dfrac{\log 1024}{\log 1016}$, or $\dfrac{\log 1024}{\log 64} = \dfrac{\log_2 1024}{\log_2 64} = \dfrac{10}{6} = \mathbf{\dfrac{5}{3}}$.

I-6. Notice that the sum of the roots is $m + n = 20$ by Viete's formulas, and $(m+n)^2 = m^2 + 2mn + n^2$. Also by Vieta's formulas, the product of the roots is $mn = -19$ and so $3mn = -57$. Because m is a root of the given equation, $m^2 - 20m - 19 = 0$. Adding, $m^2 + 2mn + n^2 + 3mn + m^2 - 20m - 19 = 2m^2 + 5mn + n^2 - 20m - 19 = 400 - 57 + 0 = 343$. This is close to the desired expression, differing by $-2n^2 + 40n + 38$, which is equal to $-2(n^2 - 20n - 19)$, which equals 0. Thus the value of $2m^2 + 5mn - n^2 - 20m + 40n + 19$ is $\mathbf{343}$.

This question is similar to Question I-6 from NYSML1994. Good math never goes bad!

I-7. Name the isosceles trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$ and $AD > BC$. Consider the diagram, which labels $AD = y$ and $BC = x$ and $CE = h$ where $\overline{CE} \perp \overline{AD}$.



The problem statement implies $AB = CD = 8$ and $AC = BD = 13$. By symmetry, $ED = \frac{1}{2}(y-x)$ and $AE = y - \frac{1}{2}(y-x) = \frac{1}{2}(y+x)$. Applying the Pythagorean Theorem to $\triangle AEC$ and $\triangle DEC$, $\left(\frac{1}{2}(y+x)\right)^2 + h^2 = 169$ and $\left(\frac{1}{2}(y-x)\right)^2 + h^2 = 64$. Subtracting the second equation from the first yields $\frac{1}{4}(y^2+2xy+x^2) - \frac{1}{4}(y^2-2xy+x^2) = \frac{1}{2}xy + \frac{1}{2}xy = xy = \mathbf{105}$.

Alternate Solution: Because isosceles trapezoids are cyclic, use Ptolemy's Theorem. Thus solve $8 \cdot 8 + BC \cdot AD = 13 \cdot 13$ to find that $BC \cdot AD = 169 - 64 = 105$.

I-8. Because $a_{10} = 0$, the first ten tosses consist of five heads and five tails, so there are $\binom{10}{5} = 252$ possible toss sequences. To solve the problem, determine how many of these sequences have both positive and negative numbers in the first ten values. There are three types of sequences: (a) those that contain only nonnegative values, (b) those that contain only nonpositive values, and (c) those that contain both positive and negative values. By symmetry, there are equal numbers of sequences of the types (a) and (b). The answer to the problem is the number of (c)-type sequences, which can be found by computing the number of (a)-type sequences and subtracting twice that value from 252. So count how many sequences contain only nonnegative values. For this to happen, the total number of heads thrown at any particular point in the sequence must be greater than or equal to the number of tails thrown.

Proceed in the following way. Consider moving a token along the grid below, starting in the upper-left corner space. Any time a coin comes up heads, move the token one space to the right. Any time a coin comes up tails, move the token one space down. Because there are five heads and five tails thrown, the token must end up in the lower-right corner space. Because the total number of heads must at all times be greater than or equal to the total number of tails, the token cannot move into any of the blacked-out spaces. The number in each space indicates the total possible number of paths that lead to that space. Because each space can only be reached from the space above or the space to the left, the total number of legal paths leading to any space equals the sum of the number of paths to the space above and the number of paths to the space to the left. Reading off the number in the bottom right indicates that there are 42 possible paths and therefore 42 possible (a)-type sequences.

1	1	1	1	1	1
	1	2	3	4	5
		2	5	9	14
			5	14	28
				14	42
					42

Therefore there are $252-42-42=168$ (c)-type sequences, hence the probability of obtaining such a sequence is $\frac{168}{252}=\mathbf{\frac{2}{3}}$.

I-9. Starting with the given equation $1+169+1849=2019$, substituting 9 for 1 yields $9+169+1849=2027$. Thus the least N greater than 2019 that can be expressed as three odd perfect squares is no greater than 2027. To establish that 2027 is indeed the desired number, consider that every odd perfect square is one more than a multiple of 8, so the sum of three odd perfect squares must be three more than a multiple of 8. The least number greater than 2019 with this property is 2027, so the answer must be **2027**.

I-10. The area of $\triangle BEA$ is $\frac{1}{2}\cdot AB\cdot AE\cdot \sin BAE=\frac{1}{2}\cdot 13\cdot AE\cdot\sin BAE$. Because $AE=15\cdot\cos CAE$, the area of $\triangle BEA$ is $\frac{1}{2}\cdot 13\cdot 15\cdot\cos CAE\cdot\sin BAE=\frac{1}{2}\cdot 13\cdot 15\cdot\frac{\sin BAC}{2}$. This is equivalent to one-half the area of $\triangle ABC$, or **42**.

Power Question 2019: Balanced Factorizations

Suppose that an integer n is factored into a product $n = a_1 a_2 a_3 \cdots a_k$. Call this factorization *balanced* if the sum of the factors is 0. That is, the factorization is balanced if $\sum_{i=1}^{k} a_i = 0$. Note that, in general, the a_i's need not be integers unless otherwise stated.

P-1. a. Find a balanced factorization of 12 into three integers. [2 pts]

b. Find a balanced factorization of 24 into four integers. [2 pts]

c. Find a balanced factorization of 240 into five integers. [2 pts]

P-2. There exists a balanced factorization of 2 into four rational numbers two of which are -4 and $\frac{1}{6}$. Find the other two rational numbers in the balanced factorization. [6 pts]

P-3. Show that there is no balanced factorization of 12 into four rational numbers, two of which are 1 and 2. [4 pts]

P-4. a. Find a balanced factorization of 12 into five not-necessarily-distinct rational numbers, three of which are integers. [2 pts]

b. Find a balanced factorization of any integer n into five not-necessarily-distinct rational numbers. [3 pts]

P-5. a. Find a balanced factorization of $28 = 7(7-3)$ into six not-necessarily-distinct rational numbers, two of which are integers. [3 pts]

b. Find a balanced factorization of any integer m of the form $m = x^2 - xy$ for nonzero integers x and y into six not-necessarily-distinct rational numbers. [3 pts]

P-6. Find a balanced factorization of 12 into 2019 not-necessarily-distinct rational numbers. [4 pts]

Let n be a positive integer. Then the set $\mathbb{Z}_n$ is defined as the set of numbers $\{0, 1, 2, \ldots, n-1\}$ where addition and multiplication are defined in the following way: $a + b$ is the result when the sum is divided by n (this is sometimes referred to as $a + b \bmod n$), and $a \cdot b$ is the result when $a \cdot b$ is divided by n (this is sometimes referred to as $a \cdot b \bmod n$). For example, in $\mathbb{Z}_5$, $3 + 4 = 2$ and $4 \cdot 4 = 1$ because the remainders when dividing 7 and 16 by 5 are 2 and 1, respectively. For an element a of the set $\mathbb{Z}_n$, define $-a$ to be the element in $\mathbb{Z}_n$ such that $a + (-a) = 0$.

P-7. Show that every number in $\mathbb{Z}_5$ has a balanced factorization into three elements of $\mathbb{Z}_5$. [5 pts]

P-8. Show that not every number in $\mathbb{Z}_4$ has a balanced factorization into three elements of $\mathbb{Z}_4$. [4 pts]

P-9. In the set $\mathbb{Z}_2$, it is possible to achieve a balanced factorization of any element into $k = 4$ elements of $\mathbb{Z}_2$ but it is not always possible to achieve a balanced factorization into $k = 5$ elements of $\mathbb{Z}_2$. Find, with proof, the values of k for which it is possible to achieve a balanced factorization of any element of $\mathbb{Z}_2$ into k elements of $\mathbb{Z}_2$. [5 pts]

P-10. Prove that if $n > 2$, it is not possible for all elements of the set $\mathbb{Z}_n$ to be broken down into a balanced factorization of 2 elements of $\mathbb{Z}_n$. [5 pts]

Solutions to 2019 Power Question

P-1. a. One such factorization is $12 = (4)(-3)(-1)$.

b. One such factorization is $24 = (1)(4)(-3)(-2)$.

c. One such factorization is $240 = (10)(-4)(-3)(-2)(-1)$.

P-2. Solve $(-4)(1/6)(x)(y) = 2$ and $-4 + \frac{1}{6} + x + y = 0$ to obtain $xy = -3$ and $x + y = \frac{23}{6}$, which implies $x\left(\frac{23}{6} - x\right) = -3 \rightarrow x^2 - \frac{23}{6}x - 3 = 0 \rightarrow 6x^2 - 23x - 18 = 0 \rightarrow (3x+2)(2x-9) = 0$, and so the solutions are the other two rational numbers in the balanced factorization: $\boxed{-\frac{2}{3} \text{ and } \frac{9}{2}}$.

P-3. The balanced factorization would have to be $12 = 1(2)(a)(-a-3)$, which implies $6 = -a^2 - 3a \rightarrow a^2 + 3a + 6 = 0$. The discriminant for this quadratic equation is $\Delta = 3^2 - 4 \cdot 1 \cdot 6 = 9 - 24 = -15 < 0$, so the quadratic equation has no real solutions, and thus there is no balanced factorization of 12 into four rational numbers, two of which are 1 and 2.

P-4. a. One such factorization is $12 = 6 \cdot 6 \cdot (-12) \cdot \frac{1}{6} \cdot \left(-\frac{1}{6}\right)$. To achieve a factorization, make the first two and last two factors multiply to -1 while the third factor is the additive inverse of the given number. To achieve the zero sum, make the first three factors add to 0 and the last two factors add to 0. This process yields the given result.

b. If $n = 0$, then $n = 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0$ is a balanced factorization of n. Otherwise, the process from **P-4a** generalizes as follows: $n = \frac{n}{2} \cdot \frac{n}{2} \cdot (-n) \cdot \frac{2}{n} \cdot \left(-\frac{2}{n}\right)$.

P-5. a. One such factorization is $28 = 7(7-3) = \frac{3}{2} \cdot \frac{3}{2} \cdot (7-3) \cdot \frac{2}{3} \cdot \left(-\frac{2}{3}\right) \cdot (-7)$. To achieve a factorization, take the additive inverse of one of the given factors, then take four factors that multiply to -1. To achieve the zero sum, make the two "reciprocal" factors have opposite signs so that they cancel, and then balance the -3 with two equal factors that add to 3. This process yields the given result.

b. The process from **P-5a** generalizes as follows:

$$m = x(x-y) = \frac{y}{2} \cdot \frac{y}{2} \cdot (x-y) \cdot \frac{2}{y} \cdot \left(-\frac{2}{y}\right) \cdot (-x).$$

P-6. Use the factorization from **P-4a** and include $2019 - 5 = 2014$ factors of 1 or -1 in equal quantities. Thus a balanced factorization would be $6 \cdot 6 \cdot (-12) \cdot \frac{1}{6} \cdot \left(-\frac{1}{6}\right) \cdot 1^{1007} \cdot (-1)^{1007}$.

P-7. The factorizations are as follows: $0 = 0 \cdot 0 \cdot 0$, $1 = 3 \cdot 3 \cdot 4$, $2 = 2 \cdot 4 \cdot 4$, $3 = 1 \cdot 1 \cdot 3$, and $4 = 1 \cdot 2 \cdot 2$.

P-8. The number 1 or the number 3 are acceptable counterexamples. Odd numbers must be products of finitely many odd numbers in $\mathbb{Z}_4$, but a sum of an odd number of odd numbers must be odd, and therefore the factorization cannot be balanced.

P-9. The answer is that a balanced factorization is possible if and only if k is even. For all even k, $a^k = a$ in $\mathbb{Z}_2$, and also $a \cdot k = 0$ in $\mathbb{Z}_2$ so the sum will vanish in this set. However, if k is odd, it is not possible to achieve a balanced factorization of 1 because an odd number of 1's add to 1 in $\mathbb{Z}_2$.

P-10. If $n > 2$, then there exists an element of $\mathbb{Z}_n$ that is not a square. To show this, consider the set of squares $\{0^2, 1^2, 2^2, \ldots, (n-1)^2\}$. These numbers have remainders when divided by n. If all of these remainders are different, then every element of $\mathbb{Z}_n$ is a square. But $1^2 = 1$ and $(n-1)^2 = n^2 - 2n + 1$ have the same remainder when divided by n. Therefore, some element y of $\mathbb{Z}_n$ is not a remainder when divided by n, and this y is not a square.

However, if an element of $\mathbb{Z}_n$ has a balanced factorization of 2 elements, that element must be a square (think: if $a = x \cdot (-x)$, then $-a$ is a square). Therefore, it is not possible to break down every element of $\mathbb{Z}_n$ into a balanced factorization of 2 elements.

Note: Those with a background in number theory can note that a residue $k \pmod{n}$ has a balanced factorization if and only if there exists an $x \in \mathbb{Z}_n$ such that $x(-x) = k$, and this is true if and only if $-k$ is a quadratic residue in $\mathbb{Z}_n$.

This Power Question is adapted from the paper Balanced Factorizations, by Anton A. Klyachko and Anton N. Vassilyev, which appeared in the MAA Monthly in December 2016.

2019 Relay Problems

R1-1. Compute the least perfect square that is greater than 2019.

R1-2. *Let N be the number you will receive.* Compute the numerical value of

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{\sqrt{N}}\right),$$

where successive denominators increase by 1.

R1-3. *Let N be the number you will receive.* Compute the number of noncongruent isosceles triangles whose side lengths are all positive integers and whose perimeter is N.

R2-1. Compute the least positive integer n for which the quantity $\sqrt{20+19\sqrt{n}}$ is also an integer.

R2-2. *Let N be the number you will receive.* Compute the least positive integer k such that

$$\sin(k \cdot N^\circ) \geq \cos(k \cdot N^\circ).$$

R2-3. *Let N be the number you will receive.* The graph of the parabola $y = x^2 - mx + N$ never goes below the line $y = m$. Compute the greatest possible value of m.

2019 Relay Answers

R1-1. 2025

R1-2. 23

R1-3. 6

R2-1. 256

R2-2. 2

R2-3. $2\sqrt{3} - 2$

2019 Relay Solutions

R1-1. Because $44^2 = 1936$ and $45^2 = 2025$, the least perfect square greater than 2019 is **2025**.

R1-2. The given expression has the value $\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \dots \cdot \frac{\sqrt{N}+1}{\sqrt{N}} = \frac{\sqrt{N}+1}{2}$. Substituting $\sqrt{N} = 45$, this has value **23**.

R1-3. The congruent sides cannot have side length $\frac{N}{4}$ or shorter, and the base must have a length that is at least 1 and that has the same parity as N. Substituting $N = 23$, the congruent sides must have length at least 6 and cannot have length greater than $\frac{23-1}{2} = 11$. But the triangles with side lengths $\{6, 6, 11\}$, $\{7, 7, 9\}$, $\{8, 8, 7\}$, $\{9, 9, 5\}$, $\{10, 10, 3\}$, $\{11, 11, 1\}$ are all noncongruent, so the answer is **6**.

R2-1. Find the least n for which $20+19\sqrt{n}$ is a square. For this to be a square, $\sqrt{n}$ must be a positive integer, say y. Now let this quantity $20+19y$ be x^2. Then $x^2 - 1 = (x-1)(x+1) = 19(y+1)$. The least x for which 19 divides $(x-1)(x+1)$ is the x for which $x+1 = 19$, i.e., $x = 18$. Then $y+1 = x-1$, so $y = 16$ and $n = y^2 = $ **256**.

R2-2. The interval in $[0°, 360°]$ for which $\sin(x) \geq \cos(x)$ is $[45°, 225°]$. Therefore, modulo 360, find the least positive integer k such that kN is within the interval $[45, 225]$. With $N = 256$, $2N = 512 = 360 + 152$, so the least such k is **2**.

R2-3. Because $x^2 - mx + N \geq m$ for all positive reals x,

$$x^2 - mx + N - m = \left(x - \tfrac{m}{2}\right)^2 + N - m - \tfrac{m^2}{4} \geq 0.$$

This is only possible if $N \geq m + \frac{m^2}{4}$. Substituting $N = 2$, and because the parabola $y = m + \frac{m^2}{4}$ slopes upward, the greatest possible value of m is the greatest root of $\frac{1}{4}m^2 + m - 2 = 0$. This root is

$$\frac{-1 + \sqrt{1^2 - 4 \cdot \frac{1}{4} \cdot (-2)}}{2 \cdot \frac{1}{4}} = \frac{-1+\sqrt{3}}{1/2} = \mathbf{2\sqrt{3} - 2}.$$

2019 Tiebreaker Problems

TB-1. A positive integer is said to be *spaced out* if every digit of the number differs from every other digit of the number by at least 2. For example, 285 and 29460 are spaced out, but 585, 10, and 374 are not. Compute the number of spaced out integers between 1000 and 9999.

TB-2. Lori divides 2019 by all integers from 1 through 2019 and records all 2019 remainders. Compute the greatest remainder that is recorded at least three times.

2019 Tiebreaker Answers

TB-1. 720

TB-2. 501

2019 Tiebreaker Solutions

TB-1. Consider first all spaced out numbers not containing a zero. To form a spaced out number not containing a zero, choose four non-consecutive numbers from the list $\{1, 2, 3, \ldots, 9\}$. Use a version of the classic "sticks and stones" technique. Consider placing stones into 4 of the 6 blanks in the sequence $\_|\_|\_|\_|\_|\_$. Then number the stones and dividing sticks from 1 through 9 ignoring any unused blanks. The digits corresponding to the stones are the digits to use. For example, the diagram $* | * | * || * |$ represents the digits 1, 3, 5, and 8 (where the $*$'s represent stones). Because it is not possible to place two stones into one blank, there must be at least one dividing stick between any two stones, so the digits will all be nonconsecutive. Reversing this process shows that every set of four nonconsecutive digits (none of which is 0) has such a diagram, so there is a one-to-one correspondence between diagrams and sets of four nonconsecutive digits. Thus there are $\binom{6}{4} = 15$ possible selections. Each selection has $4! = 24$ permutations, so there are $15 \cdot 24 = 360$ spaced out numbers not containing a zero.

Now consider spaced out numbers containing the digit 0. Such a number cannot contain a 1 but will contain three nonconsecutive digits from the list $\{2, 3, 4, \ldots, 9\}$. Using the same technique as the first case, there are $\binom{6}{3} = 20$ possible selections of three digits from the list. For each selection, the 0 cannot be in the thousands place, so there are $3 \cdot 3 \cdot 2 \cdot 1 = 18$ permutations, yielding $20 \cdot 18 = 360$ spaced out numbers one of whose digits is 0. Thus the total number of spaced out integers between 1000 and 9999 is $360 + 360 = \mathbf{720}$.

TB-2. To begin, consider the remainders obtained by dividing 2019 by each integer from 2019 down to 1. The first 1010 remainders obtained in this way are $0, 1, 2, \ldots, 1008, 1009$. Dividing 2019 by 1009 yields a remainder of 1, and the next 335 remainders are $3, 5, \ldots, 671$. Dividing 2019 by 673 yields a remainder of 0, and the next 168 remainders are $3, 6, 9, \ldots, 504$. Dividing 2019 by 504 yields a remainder of 3.
Notice that 504 is the greatest number that could appear in the list three times, but 504 is not an odd number, so it does not appear among $1, 3, 5, \ldots, 671$.
Thus 501 is the greatest number that appears in the list three times, as it is clearly in the list $0, 1, 2, \ldots, 1009$ and in the list $1, 3, 5, \ldots, 671$ and in the list $3, 6, 9, \ldots, 504$. The remainders when dividing 2019 by 503 and 502 are 7 and 11, respectively. After that, it is impossible for a division of 2019 by a number less than or equal to 501 to have a remainder of 501. Thus the greatest remainder that is recorded at least three times is **501**.

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